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THEORY OF INFINITE DERIVATIVES.*[1]

By Leon Ehrenpreis.

Introduction. We shall study here an extension of the theory of distributions of L. Schwartz (see Schwartz [1]) so as to include linear functions on spaces of analytic functions. Such an extension has been discussed from a different point of view in Ehrenpreis [1], [2]. The main viewpoint of the present paper is the following: Let R denote the real axis, and let $a = \{a_i\}$ be a sequence of complex numbers. We shall say that the indefinitely differentiable function f on R is in the domain of $(d/dx)^a$ if the series

$$\sum a_i (d^i f/dx^i)$$

converges in a certain sense. We say that the continuous function g on R is in the domain of X^a if the series

$$\sum a_i x^i g(x)$$

converges in a suitable sense. The Fourier transform (formally) maps the set of functions in the domain of $(d/dx)^a$ onto the set of functions in the domain of X^a.

Now, let A be a class of sequences of complex numbers. Call K the space of functions which are in the domain of $(d/dx)^a$ for all $a \in A$, so K is a space of analytic or quasi-analytic or non quasi-analytic functions. Call L the space of functions in the domain of X^a for all $a \in A$, so L is a space of functions which have a certain growth property at infinity. Then the Fourier transform (formally) maps K onto L. Thus, we have a characterization of the Fourier transform of classes of analytic or quasi-analytic or non-quasi-analytic functions. Conversely, the above provides a method of characterizing the Fourier transform of spaces of functions determined by growth conditions.

By means of the operators $(d/dx)^a$, X^a, we give natural topologies to the spaces K, L. K', the dual of K, is a space of continuous linear functions on a space of analytic or quasi-analytic or non quasi-analytic functions. In general, the space $\mathcal{D}$ of Schwartz (see Schwartz [1]) is not dense in K, so

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that K' cannot be identified with a space of distributions. In particular, there will not, in general, exist partitions of unity in K (or K'). However, for most of the applications of our theory to partial differential equations and to integral equations, we do not need to use partitions of unity, but rather the most important object to be studied is the Fourier transform of K'; our theory is developed specifically for this purpose.

There are two essential parts to our paper. Part I deals with spaces which are defined by both growth conditions at infinity and regularity. We study some of the subspaces of the space $\mathcal{D}$ of Schwartz [1] which are of importance in refined studies of hyperbolic partial differential equations and integral equations (see Ehrenpreis [10]). For these spaces we discuss explicitly the Fourier transform.

In part II we are concerned with spaces which are defined only by regularity conditions. We study the space of functions analytic on an arbitrary set in the plane. A topology for these spaces of analytic functions has already been given by Van Hove (see Van Hove [1], Waelbroeck [1]) but, except for some special cases, we do not know whether or not our topology is the same as the Van Hove topology.

Our notations and proofs cover only the case of dimension one, but it is easy to extend the methods to higher dimensions. In particular, we are able to give a natural definition to the Fourier transform on an analytic variety imbedded in euclidean space.

I. Spaces of infinitely differentiable functions. Let R denote the real axis, and let $a = \{a_i\}$ be a sequence of complex numbers. We shall say that the function f on R is in the domain of D^a if f lies in the space $\mathcal{S}$ of Schwartz (see Schwartz [1]) and if, for any k, the series $\sum a_i (d^i/dx^i) X^k f$ converge in the space $\mathcal{S}$, where X is the function on R: $X(x) = x$. If this is the case we set $D^a f = \sum a_i (d^i/dx^i) f$. We say that g is in the domain of X^a if $g \in \mathcal{S}$ and for any k the series $\sum a_i X^i g^{(k)}$ converges in $\mathcal{S}$. If this is the case, we set $X^a g = \sum a_i X^i g$.[2] The existence of a non-zero function g in the domain of X^a (and hence also of D^a) is clearly equivalent to the analyticity of $a(z) = \sum a_j z^j$ at $z = 0$.

[2] We have used the space $\mathcal{S}$ as the base space in our definition of infinite differentiation and infinite multiplication because we shall be most concerned with studying the behavior of functions in the domains of X^a and D^a under Fourier transform, and the Fourier transform is a topological isomorphism of $\mathcal{S}$ onto $\mathcal{S}$ (see Schwartz [1]). Actually, later on, we shall consider other types of infinite differential operator in which, for example, the use of the space $\mathcal{S}$ in the definition of D^a is replaced by that of the space $\mathcal{E}$ of Schwartz (see Schwartz [1]).

When we speak of a class C of sequences, we shall mean a class of sequences $b=\{b_i\}_{i=0}^{\infty}$ of complex numbers such that all sequences in which all but a finite number of terms are different from zero belong to C. Let $\Gamma=C_1, C_2, \cdots, L_1, L_2, \cdots$, where the C_i and L_i are classes of sequences. The space $\mathcal{G}_\Gamma$ is defined as consisting of all functions f with the following property: For any $r\geqq 1$, let a_i be a sequence in C_i and b_i a sequence in L_i for $i=1,2,\cdots,r$; then f is in the domain of $X^{b_r}D^{a_r}\cdots X^{b_1}D^{a_1}$. By the linearity of X^a and D^a we see that $\mathcal{G}_\Gamma$ (which we shall usually denote by $\mathcal{G}$ if no confusion is possible) is a vector space over the complex numbers, which may, of course, be reduced to $\{0\}$. In $\mathcal{G}$ we introduce a locally convex topology by means of the semi-norms

$$\nu_{c,d,c',d';a_1,a_2,\cdots,a_r,b_1,b_2,\cdots,b_r}(f) = \sup_{x\in R}\left|\left((d^c/dx)X^dX^{b_r}D^{a_r}\cdots X^{b_1}D^{a_1}(d^{c'}/dx)X^{d'}f\right)(x)\right|$$

for all positive integers r, c, d, c', d' and all sequence $a_i\in C_i$ and $b_i\in L_i$ for $i=1,2,\cdots,r$. Since all sequences in which only a finite number of non-zero terms appear belong to each C_i and L_i, it is immediate that $\mathcal{G}$ is Hausdorff. On the other hand, $\mathcal{G}$ need not be metrizable as a later example will show.

PROPOSITION 1. *Suppose the sets L_j, C_j have the property that, if the sequence $\{a_l\}$ belongs to any of them, then so does $\{\epsilon_l a_l\}$ whenever $|\epsilon_l|=1$ for all l. Suppose that for every k, $\{C_k{}^l a_l\}\in L_j$ (or C_j) whenever $\{a_l\}\in L_j$ (or C_j). Then $\mathcal{G}$ is complete.*

Proof. Let Q be any operator of the form $X^{b^r}D^{a^r}\cdots X^{b^1}D^{a^1}$, where, for each j, $b^j\in L_j$, $a^j\in C$, and let f lie in the closure of $\mathcal{G}$; we must show that f is in the domain of Q. An easy induction reduces the general case to the cases where $r=1$ and either D^{a^1} or $X^{b^1}=$ identity. By Fourier transform we may assume that $D^{a^1}=$ identity. Then we can find functions f^j in the domain of X^{b^1} such that $\{X^{b^1}f^j\}$ converges in the topology of $\mathcal{S}$, and, for any $l\geqq 0$. $X^lf^j\to X^lf$ in the topology of $\mathcal{S}$. Let $b^1=\{b_k\}$, $b=\{|b_k|\}$, $b'=\{(-1)^k|b_k|\}$. By our hypotheses, we may assume that $\{X^bf^j\}$ and $\{X^{b'}f^j\}$ converge in the topology of S. Let

$$B(z)=\sum|b_k|z^j, \qquad B'(z)=\sum|b_k|(-z)^k.$$

We claim first that Bf and $B'f$ are bounded functions on R. To see this, we know that $\{Bf^j\}$ converges uniformly on R. For any $x\in R$, we must have $B(x)f^j(x)\to B(x)f(x)$ because $f^j(x)\to f(x)$. Since $\{B(x)f^j(x)\}$ converges uniformly on R, it follows that $B(x)f(x)$ is uniformly bounded on R.

Call $\beta(z) = \sum |b_l||\delta_l| z^l$, where $\delta_l = C_2{}^{l+1} = l(l+1)/2$ so $\sum 1/\delta_l < \infty$, $\beta^1(z) = \sum |b_l\delta_l|(-1)^l z^l$. So βf and $\beta^1 f$ are bounded on R (and even lie in $\mathcal{S}$). Then since the coefficients of β are bounded, we can find an M so that for all $x \geqq 0$, and all l,

$$|b_l x^l \delta_l f(x)| \leqq M = \max_{x \geqq 0} |\beta(x) f(x)|,$$

that is

$$|b_l x^l f(x)| \leqq M/\delta_l.$$

This means that $\sum b_l x^l f(x)$ converges uniformly for $x \geqq 0$ (and similarly for $x \leqq 0$). Thus, $\sum b_l X^l f$ converges uniformly on R.

Now, for any k, we must show that $\sum b_l (d^k/dx^k)(X^l f)$ converges uniformly on R. We use induction on k; for $k = 0$, the result is proven above. Assume $k > 0$ and that the result is proven for derivatives of lower order. Then we have

$$\sum b_l (d^k/dx^k) X^l f = \sum b_{l,p} X^{l-p} f^{(p)},$$

where $b_{l,p} = C_p{}^{l+p} b_l$. Now, for each p, $\{b_l{}^p\} = \{b_{l+p,p}\} \in L^1$. Thus, by the result for $k = 1$, $\sum_l b_{l,p} X^{l-p} f^{(p)}$ converges uniformly on R. Thus, $\sum b_l (d^k/dx^k) X^l f$ converges uniformly on R.

By similar reasoning we find that, for any $r \geqq 0$, $k \geqq 0$,

$$\sum (1 + x^r) b^l (d^k/dx^k)(X^l f)$$

converges uniformly on R. This implies that $\sum b_l x^l f$ converges in $\mathcal{S}$. This means that f is in the domain of Q and so concludes the proof of Proposition 1.

Problem. Is it true that, for the space $\mathcal{S}$, the condition $\sum h_l$ converges in $\mathcal{S}$ implies, for any k, $\sum |(d^k/dx^k) h_l|$ converges uniformly on R? If this is the case, then the additional hpothesis $\{C_k{}^l a_l\} \in L_j$ (or C_j) is not necessary as is easily seen.

The space $\hat{\mathcal{G}}_\Gamma$ (or $\hat{\mathcal{G}}$) is defined as consisting of all functions in the domain of $D^{a_r} X^{b_r} \cdots D^{a_1} X^{b_1}$ for any positive integer r and any sequences $a_i \in C_i$, $b_i \in L_i$ for $i = 1, 2, \cdots, r$. It is clear that $\hat{\mathcal{G}}$ is a vector space over the complex numbers. The topology of $\hat{\mathcal{G}}$ is defined by means of the seminorms

$$u_{c,d,c',d';b_1 b_2, \cdots, b_r, a_1, a_2, \cdots, a_r}(f)$$
$$= |\sup_{x \in R} ((d^c/dx) X^d D^{b_r} X^{a_r} \cdots D^{b_1} X^{a_1} (d^{c'}/dx) X^{d'} f)(x)|$$

for $f \in G$, all positive integers r, c, d, c', d' and all $a_i \in C_i$, $b_i \in L_i$ for $i = 1, 2, \cdots, r$. It is clear that, given any Γ, we can find a Γ' and a Γ'' so that

$\mathcal{G}_\Gamma = \hat{\mathcal{G}}_{\Gamma'}$, $\hat{\mathcal{G}}_\Gamma = \mathcal{G}_{\Gamma''}$. Thus, the same remarks on the topology of the space $\mathcal{G}$ apply to the space $\hat{\mathcal{G}}$.

PROPOSITION 2. *G is a Montel space, that is, the closed bounded sets of G are compact.*

Proof. For every $f \in \mathcal{G}$ we associate the system $\{Qf\}$, Q being any operator of the form $X^{b_r}D^{a_r} \cdots X^{b_1}D^{a_1}$ for $a_i \in C_i$, $b_i \in L_i$. The set of these systems $\{Qf\}$ for $f \in \mathcal{G}$ may be thought of as a vector space V of continuous mappings of R into a space W which is a Cartesian product of the complex plane with itself a certain (in general, a non-denumerable) number of times. Now, if V is given the topology of uniform convergence on R (that is, a fundamental system of neighborhoods of zero in V consists of those sets N for which we can find a neighborhood M of zero in W so that N consists of those $g \in V$ with $g(x) \in M$ for all $x \in R$) then it is easily seen that $f \to \{Qf\}$ is a topological isomorphism of G onto V.

Let B be a bounded set in V. Then it is immediate that the set $\{(d/dx)Qf\}$ for $\{Qf\} \in B$ is again bounded in B. This implies that $x \to \{(Qf)(x)\}$ are, for $x \in R$, equicontinuous maps of R into W. Thus, by the theorem of Ascoli (see Bourbaki [1]) B is relatively compact (that is, the closure of B is compact) in V. Thus, V and hence $\mathcal{G}$ is a Montel space.

By similar reasoning, we could show that $\mathcal{G}$ is even a Schwartz space (see Grothendieck [1]).

By dx we shall denote the usual Lebesgue measure on R divided by 2π. Then we define the Fourier transform $\mathcal{F}$ on $\mathcal{G}$ by

$$(\mathcal{F}(f))(x) = \int f(y)\exp(-ixy)\,dy$$

for any $f \in G$, $x \in .R$, where unless otherwise specified, all integration will be taken over R. It is clear that the above integral converges uniformly for $x \in R$ and, in fact, we can easily verify from the theory of integration (see Bourbaki [2]) and the fact that the Fourier transform is a topological isomorphism of $\mathcal{S}$ onto $\mathcal{S}$:

THEOREM 1. *$\mathcal{F}$ is a topological isomorphism of $\mathcal{G}$ onto $\hat{\mathcal{G}}$.*

We shall usually use lower case letters for elements of $\mathcal{G}$ and the corresponding upper case letter for their Fourier transforms.

Let A be a topological vector space (by "vector space," we shall always mean "vector space over the complex numbers"); by A' we denote the dual of A with the topology of uniform convergence on the compact sets of A. In case that A is a Montel space, this topology is the same as that of uniform

convergence on the bounded sets of A. For $a' \in A'$, $a \in A$, we shall usually write $a' \cdot a$ for $a'(a)$.

Let B be another topological vector space and $L: A \to B$ a continuous linear or anti-linear map. Then we define the *adjoint* L' of L (see Ehrenpreis [6]) as the continuous linear (anti-linear) map of $B' \to A'$ by

$$L'b' \cdot a = b' \cdot La \qquad \text{for } L \text{ linear}$$

$$L'b' \cdot a = \overline{b' \cdot La} \qquad \text{for } L \text{ anti-linear}$$

for any $a \in A$, $b' \in B'$, where for any complex C, $\bar{C}$ is its conjugate.

The elements of $\mathcal{G}'$ (or $\hat{\mathcal{G}}'$) *are called hyperdistributions.* If Q is any operator of the form $X^{b_r}D^{a_r} \cdots X^{b_1}D^{a_1}$ with $a_j \in C_j$, $b_j \in L_j$, then Q may be thought of as defining the continuous lienar map $f \to Qf$ of $\mathcal{G} \to \mathcal{S}$. By Q' we denote the adjoint of this map.

Let us denote by Z the Banach space of functions which are continuous and bounded on R and zero at infinity with the norm $\| f \| = \max_{x \in R} |f(x)|$ for $f \in Z$. We call an element of Z' a *bounded measure.* Any $u \in Z'$ may be identified with the hyperdistribution (which we again denote by u): $f \to \int f\, du$.

PROPOSITION 3. *For any $S \in \mathcal{G}'$, we can find a finite sequence of bounded measures $u_1, \cdots, u_s$ and a corresponding sequence of operators $Q_1, \cdots, Q_s$ of the form $X^{b_r}D^{a_r} \cdots X^{b_1}D^{a_1}$ with $a_i \in C_i$, $b_i \in L_i$ such that*

$$S = \sum Q_i' u_i. \tag{1}$$

Proof. S is bounded on some neighborhood of zero in $\mathcal{G}$. Thus we can find operators $Q_1, \cdots, Q_s$ of the form as described above such that the conditions $f \in G$, $\max_{x \in R} |(Q_i f)(x)| \leqq 1$ imply $|S \cdot f| \leqq 1$. We may clearly suppose, without loss in generality, that Q_1 is the identity operator.

To each $f \in \mathcal{G}$ we associate the system $\{Q_i f\}$. Then $f \to \{Q_i f\}$ is a linear one-one map of $\mathcal{G}$ onto a set Y which is a vector subspace of Z^s (the cartesian product of Z with itself s times). Let us give Z^s the topology of a cartesian product space, and Y the topology induced by Z^s. Then the formula

$$L \cdot \{Q_i f\} = S \cdot f$$

defines L as a continuous linear function on Y. Thus, by the Hahn-Banach theorem, L can be extended to a continuous linear function T on Z^s. Since the dual of a cartesian product is the direct sum of the duals, and since the dual of Z is the space of bounded measures, we can find bounded measures $u_1, \cdots, u_s$ so that, for any $\{v_i\} \in Z^s$,

$$T \cdot \{v_i\} = \sum u_i(v_i).$$

In particular, for $f \in \mathcal{S}$,

$$\begin{aligned} S \cdot f &= L \cdot \{Q_i f\} \\ &= T \cdot \{Q_i f\} \\ &= \sum u_i \cdot Q_i f \end{aligned}$$

which is the desired result.

By c we denote the map of $\mathcal{S} \to \mathcal{S}'$: $cf \cdot g = \int \bar{f}(x) g(x)\, dx$ for any $f, g \in G$. The map ℓ is defined similarly for the space $\hat{\mathcal{S}}$. It is clear that c and ℓ are continuous. Moreover, for any $f \in \mathcal{S}$, $h \in \mathcal{S}$, $\ell h \cdot h > 0$ and $cf \cdot f > 0$ so that c and ℓ are one-one.

PROPOSITION 4. *$c(\mathcal{S})$ is dense in $\mathcal{S}'$.*

Proof. It is sufficient to show that every element of $\mathcal{S}''$ which is zero on $c(\mathcal{S})$ is zero on $\mathcal{S}'$. Let them $L \in \mathcal{S}''$ be zero on $c(\mathcal{S})$. By the theorem of Mackey and Arens (see Mackey [1], [2], Arens [1], Dieudonné and Schwartz [1]), there is a $g \in G$ so that, for any $S \in {}'\mathcal{S}$, $LS = S \cdot g$. In particular, $0 = L(cg) = cg \cdot g = \int |g(x)|^2\, dx$. Thus, $g = 0$ which means $L = 0$; this is what was to be proven.

Let us denote by $\hat{\mathcal{F}}$ the adjoint of $\mathcal{F}^{-1}$. By the usual Plancherel theorem (see Titchmarsh [2]), for any $f, g \in \mathcal{S}$,

$$\hat{\mathcal{F}} cf \cdot G = cf \cdot g = \ell \mathcal{F} f \cdot G.$$

Thus, $\hat{\mathcal{F}} cf = \ell \mathcal{F} f$ which makes it natural to call $\hat{\mathcal{F}}$ the Fourier transform on $\mathcal{S}'$. (In the terminology of Ehrenpreis [6], $\mathcal{F}$ is (c, l) unitary.) From Theorem, 1, we deduce immediately

THEOREM 2. *$\mathcal{F}$ is a topological isomorphism of $\mathcal{S}'$ onto $\hat{\mathcal{S}}'$. In fact, $\hat{\mathcal{F}}^{-1} = \mathcal{F}'$.*

We shall need several lemmas for the following examples; the first lemma is due to Poincaré [1].

LEMMA 1. *Let f be any continuous function on R. Then there exists an entire function h such that $h(x) \geqq |f(x)|$ for all $x \in R$.*

Proof. Lemma 1 will be proven if we can show the following: Given any sequence $\{a_n\}$ of positive numbers, we can find a sequence $\{h_n\}$ of non-negative numbers so that $h(z) = \sum h_n z^{2n}$ is entire and $h(j) \geqq a_j$ for all j. Let us consider the function $h(z) = \sum a_n (z/n)^{b_n}$, where the even integers b_n are chosen so large that h is entire. It is readily verified that this can be done. Then h has the desired properties and Lemma 1 is proven.

LEMMA 2. *Let $a = \{a_j\}$ be an infinite sequence with the property that $i^j a_j \geqq 0$ for all j; set $a' = \{a_j'\}$, where $a_j' = (-1)^j a_j$. Suppose that f is in the domain of D^a, and*

$$|((d^k/dx^k) D^a f)(x)| \leqq 1/16(1+x^2) \quad \text{for } k = 0, 1, 2.$$

for all $x \in R$ with a similar inequality for $D^{a'} f$ in place of $D^a f$. Then we also have $|a_j f^{(j)}(x)| \leqq 1$ for all j, and all $x \in R$.

Proof. If F denotes the inverse Fourier transform of f, then by our hypotheses we have for all $x \in R$

$$|\sum |a_j| x^j F(x)| \leqq 1/4(1+x^2)$$

and

$$|\sum |a_j| (-1)^j x^j F(x)| \leqq 1/4(1+x^2).$$

Thus, for any $x \in R$, and any j,

$$|a_j x^j F(x)| \leqq 1/4(1+x^2).$$

This gives immediately the desired result.

Definition. For any sequence $a = \{a_j\}$ for which no $a_j = 0$, we call $a^* = \{1/j! a_j\}$ the *polar sequence* of a. If h is an analytic function at 0, $h(z) = \sum h_j z^j$, with no $h_j = 0$, then $h^*(z) = \sum z^j / h_j j!$ is called the *polar polar function* of h (if it is analytic at 0).

LEMMA 3. *Given any entire function h, where $h(z) = \sum h_j z^j$, there exists an entire function h' with $h'(z) = \sum h_j' z^j$, where, for each j, $h_j' \geqq 0$, and $h_j' \geqq |h_j|$, and the polar function h'^* of h' is entire.*

Proof. For each $j \geqq 0$, set $k_j = \sup_{l \geqq j} |h_j|$; it is clear that $\{k_j\}$ is monotonically decreasing and that $K(z) = \sum k_j z^j$ is entire. Next, define $h_j' = \max(k_j, 1/\Gamma(j/2))$. Then clearly $h_j' \geqq |h_j|$ for all j. Moreover, $h_j'^* \leqq \Gamma(j/2)/j!$; it follows easily that both h' and h'^* are entire.

LEMMA 4. *Let f be any continuous function on R such that, for each $c > 0$ we can find $\eta > 0$ with $e^{c|x|} \leqq \eta f(x)$ for all $x \in R$. Then we can find an entire function h such that the polar function h^* of h is entire and such that*

$$f(x) \geqq \begin{cases} h(x) & \text{for } x \geqq 0 \\ h(-x) & \text{for } x \leqq 0. \end{cases}$$

Proof. We may clearly assume that f is monotonically increasing for $x \geqq 0$, $f(x) = f(-x)$, and that $f(x) \geqq e^x$ for all x. For each integer $l > 1$,

we choose a positive number e_l such that $f(x) \geqq 2e^{lx}$ for $x \geqq e_l$. (The existence of such an e_l is guaranteed by the hypotheses of Lemma 4.) Next, for each $l > 1$ we choose a positive integer d_l such that

$$\sum_{j \geqq d_l} l^j x^j / j! \leqq e^{(l-1)x} / z^l \text{ for } x \leqq e_l.$$

(The existence of d_l is obvious.) We define $h(z) = \sum h_j z^j$ by: $h_j = l^j/j!$ for $j = d_l, d_l + 1, \cdots, d_{l+1} - 1$, where $d_1 = 0$.

It is clear that for any l the series $\sum h_j |x|^j$ converges uniformly for $|x| \leqq e_l$, so that $h(z) = \sum h_j z^j$ is entire. That h^* is entire is obvious. Finally, for any $l > 1$, if $e_{l-1} \leqq x\, e_l$ (where $e_1 = 0$)

$$\begin{aligned} |h(x)| &\leqq \sum h_j |x|^j \\ &\leqq \sum_{j < b_l} (l-1)^j |x|^j / j! + \sum_{k \geqq l} \sum_{j \geqq b_k} k^j |x|^j / j! \\ &\leqq e^{(l-1)|x|} + \sum_{k \geqq l} e^{(l-1)|x|} / 2^k \\ &< 2e^{(l-1)|x|} \\ &\leqq f(e_{l-1}) \\ &\leqq f(x). \end{aligned}$$

This completes the proof of Lemma 4.

LEMMA 5. *Let $\{a_j\}$ be a sequence of complex numbers such that $\sum a_j z^j$ is an entire function and $i^j a_j \geqq 0$ for all j; let $f \in S$. Suppose that*

$$\max_{x \in R} (1+x^2) |\sum a_j f^{(j)}(x)| \leqq \tfrac{1}{2}, \quad \max_{x \in R} (1+x^2) |\sum (-1)^j a_j f^{(j)}(x)| \leqq \tfrac{1}{2}$$

and

$$\max_{x \in R} (1+x^2) |\sum a_j f^{(j+2)}(x)| \leqq \tfrac{1}{2}, \quad \max_{x \in R} (1+x^2) |\sum (-1)^j a_j f^{(j+2)}(x)| \leqq \tfrac{1}{2},$$

where we assume that the series $\sum a_j f^{(j)}$, $\sum (-1)^j a_j f^{(j)}$, $\sum a_j f^{(j+2)}$, $\sum (-1)^j f^{(j+2)}$ converge in $\mathcal{S}$. Then we have

$$\max_{x \in R} \sum |a_j f^{(j)}(x)| \leqq 1.$$

Proof. Call $A(z) = \sum |a_j| z^j$ and $A'(z) = \sum (-1)^j |a_j| z^j$. Then our hypotheses imply that

$$\begin{aligned} |A(x)F(x)| &\leqq \tfrac{1}{2}, \qquad |A(x)x^2F(x)| \leqq \tfrac{1}{2}, \\ |A'(x)F(x)| &\leqq \tfrac{1}{2}, \qquad |A'(x)x^2F(x)| \leqq \tfrac{1}{2}, \end{aligned}$$

for any $x \in R$. Thus,

$$(1+x^2) A(|x|) |F(x)| \leqq 1$$

for all $x \in R$. This means that

$$\max_{x \in R}(1+x^2)\sum |a_j||x|^j|F(x)| \leqq 1.$$

Now, we clearly have, for any $y \in R$,

$$|f^{(j)}(y)| \leqq \int |x^j F(x)|\,dx$$

so that

$$\begin{aligned}\sum |a_j f^{(j)}(y)| &\leqq \sum |a_j| \int |x^j F(x)|\,dx \\ &= \int \sum |a_j x^j F(x)|\,dx \\ &\leqq \int (1+x^2)^{-1}dx \\ &< 1\end{aligned}$$

because dx is the usual Lebesgue measure divided by $(2\pi)^{\frac{1}{2}}$. The interchange of summation and integration above is justied because all the terms are positive.

Remark. It is very curious that we need the Fourier transform to prove Lemma 5. Actually, it should be possible to give a direct proof of the lemma without using Fourier transform, but we have not been able to accomplish this.

LEMMA 6. *Let $l < 1$ and let β be any positive function on R such that for all $\epsilon > 0$, $\beta(x) = O(\exp(|x|^{l+\epsilon}))$. Then we can find an entire function B of order l such that $B(z)$ has positive Taylor coefficients at zero, $B(0) \geqq 1$, and $B(x) \geqq \beta(x)$ for $x \geqq 0$, and if $B(x) = \sum b_j x^j$, then $jb_j \leqq b_{j-1}$.*

Proof. Let n_0 be the smallest integer $\geqq 1$ so that $l + 1/n_0 < 1$. Choose $C \geqq 1$ so that

$$\beta(x) \leqq C\sum x^j/\Gamma(j/(l+1/n_0)+1) \text{ for } x \geqq 0.$$

(For the existence of C and the other constants to be defined below, cf. Lemma 13 below.) For each $n \geqq n_0$, we define d_n as the smallest number $\geqq 1$ (and $\geqq d_{n-1}$ for $n > n_0$) for which

$$\beta(x) \leqq \sum x^j/\Gamma(j/(l+1/n)+1) \text{ for } x \geqq d_n - 1.$$

Next we define e_{n_0} as any number $\geqq 1$ for which

$$C\sum_{j=0}^{e_{n_0}} x^j/\Gamma(j/(l+1/n_0)+1) \geqq \beta(x) \text{ for } x \leqq d_{n_0+1}.$$

We set

$$b_j = C/\Gamma(j/(l+1/n_0)+1) \text{ for } j = 0, 1, \cdots, e_{n_0}.$$

Suppose that e_n and b_j have been defined for $j \leqq e_n$ with the property that $\sum_{j=0}^{e_n} b_j x^j \geqq \beta(x)$ for $0 \leqq x \leqq d_{n+1}$. Then we choose a_{n+1} as any integer $> e_n$ for which

$$\sum_{j=0}^{e_n} b_j x^j + \sum_{j=e_n+1}^{e_{n+1}} x^j/\Gamma(j/(l+1/(n+1))+1) \geqq \beta(x)$$

for $0 \leqq x \leqq d_{n+2}$. Then we define

$$b_j = 1/\Gamma(j/(l+1/(n+1))+1) \text{ for } j = e_n+1, \cdots, e_{n+1}.$$

It is readily verified that $B(x) = \sum b_j x^j$ has the desired properties.

We shall now give several examples of spaces $\mathcal{G}$. By use of Theorem 1, we shall thus be able to characterize the Fourier transforms of certain classes of functions. The simple direct method of characterizing the Fourier transforms of the spaces in Examples 1-6 (and in more general cases) is also carried out in Hörmander [1]. However, this method does not seem strong enough to obtain the topology of $\mathcal{D}'$ in Example 1.

Example 1. We choose for L_1 the class of all sequences $\{a_j\}$ for which $\sum a_j z^j$ defines an entire function, i.e. $a_j = O(k^{-j})$ for all $k > 0$. We take $C_1, C_2, \cdots, L_2, L_3, \cdots$ as the trivial class, that is, the class consisting of all finite sequences. A function $f \in \mathcal{S}$ will belong to $\mathcal{G}$ if and only if $\sum a_j X^j f^{(l)}$ converges in $\mathcal{S}$ for any $\{a_j\} \in L_1$, and any l. Such is clearly the case if f is of compact carrier, which means that f is in the space $\mathcal{D}$ of Schwartz (see Schwartz [1]). On the other hand, if $f \in \mathcal{G}$ then the product fg is bounded on R for any entire function g. It follows easily from this that f is of compact carrier. Thus, the elements of $\mathcal{G}$ and $\mathcal{D}$ are the same.

THEOREM 3. *With the choice C_j, L_j above, the topology of $\mathcal{G}$ is the same as that of $\mathcal{D}_F$* (see Ehrenpreis [3], [4]).

Proof. For any integer $s \geqq 0$, denote by $\mathcal{D}^s$ the space of all functions of compact carrier on R which are s times continuously differentiable; $\mathcal{D}^s$ is given the usual topology (see Ehrenpreis [3], Schwartz [1]). By definition, a set N in $\mathcal{D}_F$ is a neighborhood of zero in D_F if and only if, for some $s \geqq 0$, N is the intersection with $\mathcal{D}$ with a neighborhood of zero in $\mathcal{D}^s$. It is readily verified that the topology of D^s can be described as follows: A fundamental system of neighborhoods of zero in $\mathcal{D}^s$ consists of those set N for which we can find a continuous function h on R so that N consists of all $f \in \mathcal{D}^s$ which satisfy

$$\max_{x \in R,\, 0 \leqq r \leqq s} |h(x) f^{(r)}(x)| \leqq 1. \tag{2}$$

Now (see Lemma 1), given any continuous function k on R we can find an entire function[3] p on R such that, for all $x \in R$, $|k(x)| \leqq p(x)$. Thus, for a fundamental system of neighborhoods of zero in $\mathcal{D}^s$, it is sufficient to use only entire functions for h in (2). Theorem 1 results immediately.

It follows from this and the theorem of Paley and Wiener (see Schwartz [1], Ehrenpreis [1], [3], Paley and Wiener [1]) that $\mathcal{G}$ consists of all entire functions of exponential type which are in $\mathcal{S}$. We shall now give an independent proof of this fact:

THEOREM 4. *With the choice of C_j, L_j as above, $\mathcal{G}$ consists of all entire functions of exponential type which are in $\mathcal{S}$.*

Proof. First, let $F \in \hat{\mathcal{G}}$; by Theorem 1, F is the Fourier transform of some $f \in \mathcal{G}$. Since f is of compact carrier, it follows immediately that F is an entire function of exponential type in $\mathcal{S}$.

We shall, however, give proofs of this fact which is independent of the Fourier transform. This depends on the following proposition: Let F be any function which is indefinitely differentiable on R and for which $\sum a_j F^{(j)}(x)$ converges uniformly for x in any compact set of R whenever $\{a_j\} \in L_1$; then F is an entire function of exponential type. For, if we choose $\{a_j\} = \{1/j!\}$, then it follows that, given any finite interval $I \subset R$, we can find an $M > 0$ so that, for all $x \in I$, and all j,

$$|F^{(j)}(x)| \leqq Mj! \tag{3}$$

Relation (3) implies easily (see Mandelbrojt [2]) that F is analytic on R.

We claim that, for some $l > 0$, $F^{(j)}(0) = O(l^j)$. For, assume this is is not the case; then we can find an infinite sequence of increasing indices r_j so that, for all j,

$$|F^{(r_j)}(0)| \geqq j^{r_j}.$$

Define

$$b_k = \begin{cases} j^{-k} & \text{if } k = r_j \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\{b_k\} \in L_1$, but

$$\sum b_k |F^{(k)}(0) = \infty.$$

This contradiction shows that $F^{(j)}(0) = O(l^j)$ for some $l > 0$. Since F is

[3] More precisely, we can find a function p on R which can be extended to an entire function. However, when speaking of analytic functions, we shall use the unprecise language of not distinguishing between an analytic function and its restriction to R, unless some possible confusion could result.

analytic on R, it follows immediately that F is an entire function of exponential type.

Conversely, let F be an entire function of exponential type $\leqq l$ in S. By a result of Phragmén and Lindelöf (see Titchmarsh [1], Schwartz [3]), $\exp(il\cdot)F$ is bounded in the upper half plane by $M = \max_{x \in R} |F(x)|$, where $\exp(a\cdot)$ is the function $x \to \exp(ax)$, and $\exp(-il\cdot)F$ is bounded in the lower half plane by M. For any $x \in R$, by Cauchy's formula,

$$F'(x) = \tfrac{1}{2\pi i}\int_0^{2\pi} F(e^{i\theta})e^{-2i\theta}\,d\theta \tag{4}$$

$$= \tfrac{1}{2\pi i}\int_0^{\pi} F(e^{i\theta})e^{-2i\theta}\,d\theta + \tfrac{1}{2\pi i}\int_{\pi}^{2\pi} F(e^{i\theta})e^{-2i\theta}\,d\theta.$$

The first integral in (4), in modulus, does not exceed

$$\tfrac{1}{2\pi}\int_0^{\pi} M\,|\exp(-ile^{i\theta})|\,d\theta = (M/2\pi)\int_0^{\pi}\exp(l\sin\theta)\,d\theta.$$

Similarly,

$$|\tfrac{1}{2\pi i}\int_{\pi}^{2\pi} F(e^{i\theta})e^{-2i\theta}\,d\theta| \leqq (M/2\pi)\int_{-\pi}^{0}\exp(-l\sin\theta)\,d\theta.$$

Since, clearly,

$$\int_0^{\pi}\exp(l\sin\theta) = \int_{-\pi}^{0}\exp(-l\sin\theta)\,d\theta \leqq \pi e^l,$$

we have, for any $x \in R$,

$$|F'(x)| \leqq Me^l. \tag{5}$$

By iteration, for any n and any $x \in R$,

$$|F^{(n)}(x)| \leqq Me^{ln}. \tag{6}$$

Hence, by the above-stated Phragmén-Lindelöf theorem, for any z with $|\Im(z)| \leqq 1$, and any n, ($\Im(z)$ denotes the imaginary part of z)

$$|F^{(n)}(z)| \leqq Me^{l(n+1)}. \tag{7}$$

It results immediately from (7) that, for any sequence $\{a_n\} \in L_1$, $\sum a_n F^{(n)}(z)$ converges uniformly for $|\Im(z)| \leqq 1$. Now, for any $\{a_n\} \in L_1$, we have

$$\sum_{n=0}^{r} a_n zF^{(n)}(z) = \sum_{n=0}^{r-1} a_n' F^{(n)}(z) + \sum_{n=0}^{r} a_n (FX)^{(n)}(z), \tag{8}$$

where $a_n' = (n+1)a_{n+1}$, as follows from the commutation relation between multiplication by X and $(d/dx)^j$:

$$(FX)^{(j)} - XF^{(j)} = jF^{(j-1)}$$

which is easily proved by induction. Now, for $\{a_n\} \in L_1$ $\{a_n'\}$ is again in L_1; moreover, FX is again an entire function of exponential type in $\mathcal{S}$. It follows from (8) and our previous remarks that $\sum a_n z F^{(n)}(z)$ converges uniformly for $|\mathfrak{I}(z)| \leqq 1$. By iteration, for any j, $\sum a_n z^j F^{(n)}(z)$ converges uniformly in $|\mathfrak{I}(z)| \leqq 1$. By Cauchy's formula, this implies that, for any j and any k, $\sum((d^k/dx^k)a_n X^j F^{(n)})(x)$ converges uniformly in R. By the definition of the topology of $\mathcal{S}$ it follows that $\sum a_n F^{(n)}$ converges in $\mathcal{S}$. This completes of the proof of Theorem 4.

The topology of $\hat{\mathcal{G}} = \hat{\mathcal{D}}_F$ is described in Ehrenpreis [3]. We give here first another expression for the topology of $\hat{\mathcal{D}}_F$ and we show later that the two topologies are the same.

THEOREM 5. *Let a be any sequence such that $A(z) = \sum a_j z^j$ and $A^*(z) = \sum a^*_j z^j$ define entire functions of greater than exponential type; assume in addition that $i^j a_j > 0$ for all j. Let r be an integer and consider the set of $F \in \hat{\mathcal{D}}_F$ which satisfy, for $s, k, p = 0, 1, 2, \cdots, r$*

$$\max_{\mathcal{J}(z)=y} |z^p (d^s/dz^s) F(z)| \leqq \begin{cases} A(iy) & \text{for } y \geqq 0 \\ A(-iy) & \text{for } y \leqq 0, \end{cases} \tag{9}$$

where $y = \mathfrak{I}(z)$. Then the sets N form a fundamental system of neighborhoods of zero in $\mathcal{D}_F$.

Proof. The fact that $A^*(z)$ is an entire function means that, for any l, $|a_j| \geqq l^j/j!$ for all but a finite number of l. This means that the sets N defined above have the property that, for any $F \in \hat{\mathcal{D}}_F$, there is a $c > 0$ such that $cF \in N$. From this we deduce easily that the sets N define a locally convex topology on the set of functions of $\hat{\mathcal{D}}_F$.

That each N is a neighborhood of zero in $\mathcal{D}_F$ follows immediately from the fact that (see Lemma 2 above) conditions (9) are implied by

$$\max_{x \in R} |(d^p/dx^p) D^a X^k F(x)| \leqq 1/16(1 + x^2) \tag{10}$$

for $p = 0, 1, 2$ and $k = 0, 1, 2, \cdots, r$.

To conclude the proof of Theorem 5 we must show that, for M any neighborhood of zero in $\hat{\mathcal{D}}_F$, there exists an N as above with $N \subset M$. We may suppose that M is defined by an entire function $H(z) = \sum h_j z^j$ and an integer r so that M consists of all $F \in \hat{\mathcal{D}}_F$ such that,

$$\max_{x \in R} |(d^s/dx^s) X^k D^h X^p F(x)| \leqq 1 \tag{11}$$

for $k, s, p = 0, 1, 2, \cdots, r$.

For each integer $l > 0$ we choose a positive number c_l so that $c_1 = 3$, $c_l < c_{l+1}$, and

$$h_l e^l / c_l{}^l \leqq 1/2^l.$$

Next we define the function g on the imaginary axis by $g(iy) = g(-iy) = e^l$ for $c_l + 1 \leqq y \leqq c_{l+1} + 1 - \epsilon$ (for some $\epsilon > 0$) and g is so defined as to be continuous and monotonic.

By Lemma 4 we can find an entire function H whose polar function H^* is entire such that $H(iy)$ has positive coefficients in its Taylor expansion and $H(iy) \leqq g(iy)$ for all y. Call N_0 the set of $F \in \hat{\mathcal{D}}_F$ for which

$$\sup_{\mathcal{J}(z)=y, y \geqq 2} |(d^s/dx^s) X^p F(x)| \leqq H(i|y|)/16(1+z^2)(2r)! \tag{12}$$

for $s, p = 0, 1, 2, \cdots, 2r$. By Cauchy's formula, for any $F \in N_0$, $F(z) = \sum f_j z^j$, we have

$$\begin{aligned} |h_j f_j| &= \left| \int_{ic_j-}^{ic_j+} z^{-(j+1)} F(z)\, dz \right| |h_j| \\ &\leqq |h_j|\, e^j / c_j{}^j r! \\ &\leqq 1/2^j r!. \end{aligned}$$

Thus, $(D^h F)(0) \leqq 1/(2r)!$. Similarly, for any z with $|\mathcal{J}(z)| \leqq 1$, we have $|D^h F(z)| \leqq 1/(2r)!$. Applying the above to $X^p F$ in place of F ($p = 0, 1, 2, \cdots, 2r$) we deduce that $|D^h X^p F(z)| \leqq 1/(2r)!$ for $p = 0, 1, 2, \cdots, 2r$, and for $|\mathcal{J}(z)| \leqq 1$.

For $k = 0, 1, 2, \cdots, r$ let $a = \{a_j{}^k\}$ be defined by

$$a_j{}^k = (j+1)(j+2)\cdots(j+k) a_{j+k}.$$

For each $k = 0, 1, 2, \cdots, r$ we determine a set N_k as above corresponding to the sequence a_k just as the set N_0 was determined corresponding to the sequence $a = a^0$. Then, by the above and induction on (8) it follows that

$$M \supset ((1/r!)N_0 \cap (1/r!)N_1 \cap \cdots \cap (1/r!)N_r).$$

This completes the proof of Theorem 5.

As an immediate consequence of Theorem 5, Lemma 4, and Cauchy's formula, we have

COROLLARY. *Let g be any continuous function on R such that, for every $c \geqq 0$, we can find an $\eta > 0$ such that $e^{c|x|} \leqq \eta g(x)$ for all $x \in R$. Let r be an integer and call N the set of $F \in \hat{\mathcal{D}}_F$ which satisfy*

$$\max_{I(z)=y} |z^p F(z)| \leqq f(y) \tag{12}$$

for all y and for $p = 0, 1, 2, \cdots, r$. Then the sets N form a fundamental system of neighborhoods of zero in $\hat{\mathcal{D}}_F$.

Remark. From the above Corollary it follows easily that the topology of $\hat{\mathcal{D}}_F$ described above is the same as that described in Ehrenpreis [3].

Example 2. Let l be a positive number. L_1 consists of all sequences $\{a_n\}$ for which there is an $\epsilon > 0$ so that $\sum a_n z^n$ is analytic in the circle, center origin, radius $l + \epsilon$. $L_2, L_3, \cdots, C_1, C_2, \cdots$ are defined to be the trivial sets. It is easily seen that $\mathcal{G}$ is just the space $\mathcal{D}_l$ (see Ehrenpreis [1], [3], Schwartz [1]) of indefinitely differentiable functions which vanish for $|x| \geqq l$. The topology of $\mathcal{G}$ is the same as that of $\mathcal{D}_l$; this is a metrizable space in which a sequence $\{f_j\}$ converges to zero if and only if, for any k, $f_j^{(k)}(x) \to 0$ uniformly for $x \in R$.

Thus, $\hat{\mathcal{G}}$ is the space $\hat{\mathcal{D}}_l$ of all entire functions of exponential type $\leqq l$ which are in $\mathcal{S}$ (see Ehrenpreis [1], [3]). A sequence $\{F^j\}$ converges to zero in $\hat{\mathcal{D}}_l$ if and only if, for any k, $(1 + |z|^k)F^j(z) \to 0$ uniformly in $|\Im(z)| \leqq 1$. The method of proof of Theorem 4 is not quite strong enough to establish this characterization of $\hat{\mathcal{G}}$ independently of the method of Ehrenpreis [3]. However, it was shown by Bernstein (see Schaeffer [1]) that the inequality (5) can be improved to

$$|F(x)| \leqq Ml \quad \text{for all } x \in R \tag{5'}$$

where, as in (5), F is an entire function of exponential type $\leqq l$ which is bounded by M on R. Using (5') instead of (5) we can prove the above characterization of $\hat{\mathcal{G}}$ by the method of proof of Theorem 4. The direct description of the topology of $\hat{\mathcal{D}}_l$ can also be obtained by use of (a simplified form of) the method of proof of Theorem 5.

Example 2 provides us with an example in which the space $\mathcal{G}$ is metrizable (although this situation does not seem to occur often).

Example 3. Let l be a positive number. L_1 consists of all sequences $\{a_n\}$ for which $\sum a_n z^n$ is analytic on the open circle, center origin, radius l. $L_2, L_3, \cdots, C_1, C_2, \cdots$ are the trivial sets.

The methods used to determine the spaces $\mathcal{G}$, $\hat{\mathcal{G}}$ are exactly the same as in Example 1. We can show that $\mathcal{G}$ consists of all indefinitely differentiable functions which vanish outside of some compact subset of the interval $[-l < x < l]$. The space $\mathcal{G}$ is topologically isomorphic to the space $\mathcal{D}_F$; the isomorphism can be given, e. g., by $f \in G \leftrightarrow h \in D_F$ where

$$f(x) = \begin{cases} h(\tan(\pi x/2)l)) & \text{for } |x| < l \\ 0 & \text{for } |x| \geqq l. \end{cases}$$

Example 4. Let l be a positive number. L_1 consists of all sequences $\{a_j\}$ for which $|\sum a_j z^j| = O(e^{l|z|})$, or, what is the same thing, $a_j = O(l^j/j!)$. $L_2, L_3, \cdots, C_1, C_2, \cdots$ are defined to consists of the finite sequences only.

It is readily verified that $\mathcal{G}$ is the space of all functions $f \in \mathcal{S}$ for which $\exp(l\cdot)f$ and $\exp(-l\cdot)f$ are again in $\mathcal{S}$. $\mathcal{G}$ is a metrizable space; a sequence $\{f_n\}$ converges to zero in $\mathcal{G}$ if and only if

$$\exp(l\cdot)f \to 0 \text{ in } \mathcal{S}, \qquad \exp(-l\cdot)f \to 0 \text{ in } \mathcal{S}.$$

$\hat{\mathcal{G}}$ consists of all functions F which are analytic in the strip $-l < \Im(z) < l$ and continuous on the boundary with the additional properties

(a) Whenever $-l \leqq c \leqq l$, the function $x \to F(x+ic)$ lies in S.

(b) The set $\{x \to F(x+ic)\}_{-l \leqq c \leqq l}$ is bounded in S.

(By use of a generalization of the Phragmén-Lindelöf principle, it follows that conditions (a) and (b) are implied by much less stringent conditions on F; see Ehrenpreis [2].) A sequence $\{F_n\}$ converges to zero in $\hat{\mathcal{G}}$ if and only if, for any integer r, the sequences $x \to F_n(x \pm il)$ converge to zero in $\mathcal{S}$. This example as well as Example 5 below have also been considered by Schwartz [5] and Lions [1].

The proof of the above statements about the characterization and topology of $\hat{\mathcal{G}}$ can be proven in a simple manner by use of the Fourier transform. Thus, any $F \in \hat{\mathcal{G}}$ is the Fourier transform of an $f \in \mathcal{G}$ from which it follows immediately that F satisfies conditions (a) and (b) above. On the other hand, if F satisfies conditions (a) and (b), then, if f denotes the inverse Fourier transform of F, we have by Cauchy's theorem,

$$f(x) = e^{-lx} \int_{-\infty}^{\infty} F(t+il) e^{ixt}\, dt$$

$$f(x) = e^{lx} \int_{-\infty}^{\infty} F(t-il) e^{ix}\, dt.$$

It follows from the fact that the Fourier transform is a topological isomorphism of $\mathcal{S}$ onto $\mathcal{S}$ that $f \in \mathcal{G}$. By similar methods we can verify the above statement on the topology of $\hat{\mathcal{G}}$.

We can also give direct proofs of the characterization of $\hat{\mathcal{G}}$ and its topology without use of the Fourier transform.

Example 5 . Let l be a positive number. L_1 consists of all sequences

$\{a_j\}$ for which $A(z) = \sum a_j z^j$ is an entire function of exponential type $\leqq l$, that is, for every $\epsilon > 0$, $A(z) = O(e^{(l+\epsilon)|z|})$. This is the same as saying that

$$\limsup(\log(j!\,|\,a_j\,|)/j) = \log l. \tag{13}$$

We write $\mathcal{K}_l$ for $\mathcal{G}$ and $\hat{\mathcal{K}}_l$ for $\hat{\mathcal{G}}$.

We have

THEOREM 6. *$f \in \mathcal{K}_l$ if and only if, for each r we can find an $\epsilon > 0$ such that $f^{(r)}(x) = O(e^{-(l+\epsilon)|x|})$.*

COROLLARY. *A function F on R is in $\hat{\mathcal{K}}_l$ if and only if, for each integer n, we can find an $\epsilon_n > 0$ so that $Z^n F$ is bounded and analytic in the strip $|\mathfrak{I}(z)| \leqq l + \epsilon_n$.*

The above Corollary can be deduced easily from Theorem 6 by means of the Fourier transform; a direct proof can also be given.

Example 6. Let l be a positive number. L_1 consists of all sequences $\{a_j\}$ for which $A(z) = \sum a_j z^j$ is an entire function of order $\leqq l$, that is, for every $\epsilon > 0$, $A(z) = O(\exp(|z|^{l+\epsilon}))$. This is the same as saying (see Titchmarsh [1], p. 253) that

$$\liminf(\log(1/|a_j|)/j\log j) = 1/l. \tag{16}$$

We shall write $\mathcal{A}_l$ for $\mathcal{G}$ and $\hat{\mathcal{A}}_l$ for $\hat{\mathcal{G}}$.

THEOREM 7. *Let $f \in \mathcal{S}$; a necessary and sufficient condition that $f \in \mathcal{A}_l$ is: For each k we can find an $\eta > 0$ such that*

$$f^{(k)}(x) = O(\exp(-|x|^{l+\eta}).$$

Proof. The sufficiency of the condition is readily verified; we proceed to the necessity. Let $f \in \mathcal{A}_l$ and assume that, for some k and for no $\eta > 0$ is $f^{(k)}(x) = O(\exp(-|x|^{l-\eta})$. We suppose that $k = 0$; the general case is handled similarly.

By our assumption, for any $n > 0$ we can find a point $c_n \in R$ such that $f(c_n) \geqq \exp(-|c_n|^{l+1/n})$, and the $|c_n| \to \infty$. We shall construct a function H which is an entire function of order l and such that $H(c_n) \geqq \exp(|c_n|^{l+1/n})$. Thus, if $H(z) = \sum h_j z^j$, it is clear that f is not in the domain of X^h.

The construction of H proceeds along essentially the same lines as the proof of Lemma 1. We assume that all the c_n are positive and that $1 < c_n < c_{n+1}$ for all n; the general case is easily reduced to this case or to the case where all c_n are negative and $1 > c_n > c_{n+1}$ for all n, which is handled similarly. We set

$$H(z) = \sum_{j=0}^{\infty} \exp(c_n{}^{l+1/n})\,(z/c_n)^{e_n}, \tag{17}$$

where the integers e_n will be chosen so that H is an entire function of order l. Since $H(c_n) > \exp(c_n{}^{l+1/n})$, this will complete our proof. By (17) H will be of order $\leqq l$ if

$$\log(c_n{}^{e_n}/\exp(c_n{}^{l+1/n}))/e_n \log e_n \geqq 1/(l+1/n) - \epsilon_n \text{ for all } n, \tag{18}$$

where $\epsilon_n \to 0$. The left side of (18) is

$$[e_n \log c_n - c_n{}^{l+1/n}]/e_n \log e_n = \log c/\log e_n - c_n{}^{l+1/n}/e_n \log e_n.$$

We choose $e_n = [c_n{}^{l+1/n}]$ where, for $x \in R$, $[x]$ is the greatest integer $\leqq x$. We may clearly assume that the c_n are so chosen that $e_n < e_{n+1}$ for all n. Then the left side of (18) is $\geqq 1/(l+1/n) - \epsilon_n$, where we may choose $\epsilon_n = e_n + 1/e_n \log e_n$ so that $\epsilon_n \to 0$. This completes the construction of H and so completes the proof of Theorem 7.

Theorem 7 shows that the space in Example 6 is the same as a space considered by Gelfand and Silov [1].

We can also prove, either directly or by means of Fourier transform,

THEOREM 8. *Let $l \geqq 1$ and let F be an entire function in $\mathcal{S}$. A necessary and sufficient condition that $F \in \hat{\mathcal{A}}_l$ is: For any integer $n \geqq 0$ we can find an $\eta' > 0$ so that*

$$\sup_{\mathcal{J}(z)=y} |z^n F(z)| = O(\exp(|y|^{l'-\eta'})), \tag{19}$$

where l' is the conjugate exponent of l, i.e. $1/l + 1/l' = 1$.

THEOREM 9. *Let h be any continuous positive function on R such that, for every $\epsilon > 0$, $\exp(-|x|^{l+\epsilon}) = O(h(x))$; let m be an integer. Call N the set of $f \in A_l$ which satisfy, for all $x \in R$,*

$$|f^{(k)}(x)| \leqq h(x) \text{ for } k = 0, 1, 2, \cdots, m. \tag{22}$$

Then these sets N form a fundamental system of neighborhoods of zero in A_l.

THEOREM 10. *Let $l \geqq 1$; let g be any continuous function on R such that, for every $\epsilon > 0$, $\exp(|x|^{l'-\epsilon}) = O(g(x))$; let m be a positive integer. Call M the set of $F \in \hat{\mathcal{A}}_l$ such that*

$$\sup_{\mathcal{J}(z)=y} |z^k F(z)| \leqq g(y) \text{ for } k = 0, 1, \cdots, m. \tag{24}$$

Then the sets M form a fundamental system of neighborhoods of zero in $\hat{\mathcal{A}}_l$.

Example 7, non quasi-analytic classes. Up to now, we considered spaces $\mathcal{S}$ and $\hat{\mathcal{S}}$ for which only one of the C_i, L_i was non-trivial. For the next example, we choose L_1 as consisting of all sequences $\{a_j\}$ for which $\sum a_j z^j$ is an entire function. C_1 consists of all sequences $\{b_j\}$ such that $\sum b_j z^j$ is an

entire function of order l, where $l<1$ is fixed. C_j and L_j for $j>1$ are defined to be trivial sets. We write $\mathfrak{J}_l$ for $\mathfrak{S}$ and $\hat{\mathfrak{J}}_l$ for $\hat{\mathfrak{S}}$.

The space $\mathfrak{J}_l$ consists of those functions f of compact carrier which satisfy

$$f^{(k)}(x)=O(k^{k/l'}) \text{ uniformly in } x \tag{27}$$

for some $l'>l$. This follows readily from equation (16). Thus, if $l\geqq 1$, f would have to be analytic, so $\mathfrak{J}_l$ would consist of $\{0\}$ only. If $l<1$, it is well-known (see Paley and Wiener [1], Mandelbrojt [2]) that $\mathfrak{J}_l$ is non-empty. We shall see later that, in fact, $\mathfrak{J}_l$ is dense in $\mathfrak{D}$.

For the use of the spaces $\mathfrak{J}_l$ in partial differential equations, see e.g. Ehrenpreis [7], [10], [11], [12], Hörmander [1], [3], Friedman [1].

Using Theorem 4 and methods similar to the above, we can deduce

THEOREM 11. *$\hat{\mathfrak{J}}_l$ consists of all entire functions F of exponential type which satisfy*

$$F(x)=O(\exp(-|x|^{l+\eta})) \tag{28}$$

for some $\eta>0$.

We can now prove

THEOREM 12. *For any $l<1$, $\mathfrak{J}_l$ is dense in $\mathfrak{D}$.*

Proof. It follows from the above Theorem 11 that $\mathfrak{J}_l$ is an (algebraic) ideal in $\mathfrak{D}$ under convolution. Moreover, since $\mathfrak{J}_l$ is non empty, we can find a function $f\in\mathfrak{J}_l$ which is not identically zero. Then if c is any zero of F with order r, it follows from Theorem 11 that $F(z)/(z-c)^r$ is again in $\mathfrak{J}_l$. Thus, the (algebraic) ideal $\hat{\mathfrak{J}}_l$ has no common zeros. It follows from the fundamental theorem of mean periodic functions (see Schwartz [4], Ehrenpreis [5]) that $\mathfrak{J}_l$ is dense in $\mathfrak{D}$, which is the desired result.

Remark. By a similar method we could show that $\mathfrak{J}_l$ is dense in $\mathfrak{J}_{l'}$ whenever $l>l'$.

We consider first the topology of $\mathfrak{J}_l$:

THEOREM 13. *Let γ be any continuous positive function on C such that, for any ϵ, k, we have* ($z=x+iy$)

$$\exp(-|x|^{l+\epsilon}+k|y|)=O(\gamma(x,y)), \tag{29}$$

and γ is the product of a function of x by a function of y. Let N be the set of $F\in\hat{\mathfrak{J}}_l$ which satisfy

$$|F(z)|\leqq\gamma(z) \text{ for all } z\in C.$$

Then these sets N form a fundamental system of neighborhoods of zero in $\hat{\mathfrak{J}}_l$.

Proof. Let us note the following: A set $M \subset \hat{\mathfrak{J}}_l$ is a neighborhood of zero if and only if we can find a neighborhood of zero P in $\mathfrak{D}_F$ and entire functions $B_1, B_2, \cdots, B_r$ of order l such that M consists of all $F \in \hat{J}_l$ for which $B_j F \in P$ for $j = 1, 2, \cdots, r$. This follows readily from the definition of the topologies of $\hat{D_F}$ and J_l by means of infinite derivatives.

Now, by Theorem 5, there are a continuous function A on R and an integer s so that P consists of all $G \in D_F$ which satisfy $(z = x + iy)$

$$|z^p G(z)| \leqq A(y)$$

for $p = 1, 2, \cdots, s$, and for all $z \in C$; A is a positive function with the property that $\exp(k|y|) = O(A(y))$ for all k. Thus $F \in \hat{\mathfrak{J}}_l$ is in M if and only if, for $j = 1, 2, \cdots, r$, $p = 1, 2, \cdots, s$, we have

$$|z^p B_j(z) F(z)| \leqq A(y) \tag{30}$$

for all $z \in C$.

Let us set

$$\alpha(w) = \max_{j=1,2,\cdots,r,\, p=1,2,\cdots,s,\, |z|=2w} |z^p B_j(z)|,$$

so α is of order l, that is, $\alpha(w) = O(\exp(|w|^{l+\epsilon}))$ for every $\epsilon > 0$. We define

$$\gamma(z) = A(y)/[1 + \alpha(|y|)][1 + \alpha(|x|)]. \tag{31}$$

Then it is readily verified that γ satisfies the hypotheses of Theorem 13.

If $F \in \hat{\mathfrak{J}}_l$ satisfies $|F(z)| \leqq \gamma(z)$ for all z, then we have, for any j, p,

$$|z^p B_j(z) F(z)| \leqq |z^p B_j(z)|\, A(y)/[1 + \alpha(|y|)][1 + \alpha(|x|)].$$

Now, either $|y|$ or $|x|$ is $\geqq \frac{1}{2}|z|$ so that

$$|z^p B_j(z)|/[1 + \alpha(|y|)][1 + \alpha(|x|)] \leqq 1.$$

Thus the condition $|F(z)| \leqq \gamma(z)$ implies

$$|z^p B_j(z) F(z)| \leqq A(y)$$

for $p = 1, 2, \cdots, s$, $j = 1, 2, \cdots, r$ and all y. This means that $F \in M$.

We have shown that every neighborhood of zero in $\mathfrak{J}_l$ contains a set N as described in the statement of our theorem. It remains to show that these sets N are neighborhoods of zero in J_l. To this end, let γ be any function satisfying (31). We set

$$\beta(x) = \gamma(x, 0). \tag{32}$$

Then it follows from (31) that, for any $\epsilon > 0$,

$$\exp(-|x|^{l+\epsilon}) = O(\beta(x)). \tag{33}$$

Next, we choose a fixed $\eta > 0$ so that $l + \eta < 1$. We set

$$A(2y) = \gamma(0, y)\exp(-|3y|). \tag{34}$$

It is clear that, for any k,

$$\exp(k|y|) = O(A(y)).$$

We may assume, without any loss in generality, that β is even and monotonically decreasing and that A is even and monotonically increasing.

Next (see (33) and Lemma 6 above), we choose an entire function B of order l such that $B(z+1)$ has positive Taylor coefficients at zero, $B(-1) \geqq 1$, and $B(x) \geqq 1/\beta(x+1)$, for $x \geqq -1$, and $|B(x)/B(x-1)|$ is bounded for $x \geqq 0$. Let M be the set of $F \in \hat{\mathcal{J}}_l$ which satisfy

$$|F(z)B(z)| \leqq A(y), \qquad |F(z)B(-z)| \leqq A(y),$$

for all $z = x + iy$; by what we have said above, M is a neighborhood of zero in $\hat{\mathcal{J}}_l$. We claim that, for a suitable choice of $c > 0$, $cM \subset N$.

For this purpose, we have to note that the function B could be modified slightly so that we can assume that B is large in some angular sector containing the positive real axis. For, we can easily extend Lemma 13 below so as to mawe an inequality like (70):

$$\alpha|\mathcal{R}Q(z)| \leqq |\mathcal{R}\exp(z^r)| \leqq \beta|\mathcal{R}Q(2z)| \tag{70'}$$

hold in a sufficiently small angular sector containing the positive real axis. ($\mathcal{R}$ denotes the real part.) Then the proof of Lemma 6 can be modified to show (using the minimum modulus theorem 5 of [5] to handle small values of x) that we can assume B has the property stated in:

LEMMA 7. *We can find a constant d so that, given any y_0 we can find a y' lying between y_0 and $2y_0$ such that for any $x \geqq 0$,*

$$B(x + iy') \geqq d\exp(-|y'|)/\beta(x). \tag{35}$$

Moreover

$$B(x + iy') \geqq d\exp(-|y'|)|B(x-1)|. \tag{36}$$

Thus, any $F \in M$ satisfies, for $x \geqq 0$,

$$\begin{aligned} |F(x+iy')| &\leqq A(y')/|B(x+iy')| \\ &\leqq dA(y')\beta(x)\exp(|y'|) \\ &\leqq dA(2y_0)\beta(x)\exp(|3y_0|) \\ &= d\gamma(x, y_0). \end{aligned} \tag{37}$$

Now, inequality (37) is not good enough because it is only proven for some y' lying between y_0 and $2y_0$. In order to complete the proof of our theorem, we consider $B(z-iy_0)F(z)$ instead of $F(z)$. We have, for all $x \geqq 0$, (see Lemma 8 below) by Lemma 7,

$$(38)\quad \begin{aligned} |B(x+iy'-iy_0)F(x+iy')| &\leqq B(x)\exp(|y'-y_0|)|F(x+iy')| \\ &\leqq B(x)\exp(|y_0|)|F(x+iy')| \\ &\leqq (1/d)(B(x)/|B(x-1)|)\exp(|y_0|)A(2y_0)\exp(|2y_0|). \end{aligned}$$

By Lemma 8 below, $|B(x)/B(x-1)| \leqq e$ for all $x \geqq 0$. Thus,

$$(39)\qquad |B(x+iy'-iy_0)F(x+iy')| \leqq (e/d)\exp(|3y_0|)A(2y_0)$$

for all $x \geqq 0$ and similarly for all $x \leqq 0$. Similarly, we can find a y'' lying between $-y_0$ and $-2y_0$ for which an inequality similar to (39) holds. Thus, by the maximum modulus theorem applied to the strip lying between y' and y'', we have

$$(40)\qquad |B(|x|)F(x+iy_0)| \leqq (e/d)\exp(|3y_0|)A(2y_0),$$

or

$$\begin{aligned} |F(x+iy_0)| &\leqq (e/d)\exp(|3y_0|)A(2y_0)/B(|x|) \\ &\leqq (e/d)\exp(|3y_0|)A(2y_0)\beta(x) \\ &\leqq (e/d)\gamma(x,y_0). \end{aligned}$$

Thus, if we choose $c = d/e$, we have $cF \in N$. This proves that $cM \subset N$ which concludes the proof of Theorem 13.

We have used in the proof of Theorem 13

LEMMA 8. *Let B, η be as above; then we have*

$$|B(x+y)| \leqq B(x)\exp(|y|)$$

for all $x \geqq -1$ and all y (real or complex).

Proof. We write $B(x_0) = \sum b_j(x_0+1)^j$. Then if $x_0 \geqq -1$, the Taylor coefficients of B at x_0 are

$$(41)\qquad b_k(x_0) = (1/k!)\sum_{j=0}^{\infty} j(j-1)\cdots(j-k)(x_0+1)^{j-k}b_j.$$

By our construction of B (see Lemma 6), we know that, for all j, we have $jm_j \leqq b_{j-1}$, so that $j(j-1)\cdots(j-k)b_j \leqq b_{j-k}$. This gives, by (41),

$$b_k(x_0) \leqq (1/k!)\sum_{j=0}^{\infty} b_{j-k}(x_0+1)^{j-k} = B(x_0)$$

because all the b_t are positive. Thus, for $x \geqq -1$,

$$(42) \qquad \begin{aligned} |B(x+y)| &\leqq B(x+|y|) \\ &= \sum b_k(x)|y|^k \\ &\leqq B(x)\sum |y|^k/k! \\ &= e^{|y|}B(x). \end{aligned}$$

This is the desired result.

The topology of $\mathfrak{J}_l$ is very closely related to that of $\mathfrak{D}_F$. In fact, we have

THEOREM 14. *The topology of $\mathfrak{J}_l$ can be described as follows:*

(1) *Let $a(z) = \sum a_j z^j$ be an entire function of order l, and let b be an entire function. Let N be the set of $f \in \mathfrak{J}_l$ for which*

$$(43) \qquad \max_{x \in R} |b(x)| \sum |a_j^{(j)}(x)| \leqq 1.$$

Then the sets N form a fundamental system of neighborhoods of zero in $\mathfrak{J}_l$.

(2) *Let $c = \{c_j\}$ be any sequence of positive numbers such that, for any $l' > l$, we have*

$$(44) \qquad j^{j/l'} = O(c_j).$$

Let e be an entire function, and let M be the set of $f \in \mathfrak{J}_l$ which satisfy, for all $x \in R$,

$$(45) \qquad |f^{(j)}(x)| \leqq c_j |e(x)|.$$

Then these sets M constitute a fundamental systems of neighborhoods of zero in $\mathfrak{J}_l$.

Proof. (1) It is easily seen from the definitions that every neighborhood of zero in $\mathfrak{J}_l$ is contained in some N. Thus, we have to show that the sets N are neighborhoods of zero. This is not at all obvious because the topology of $\mathfrak{J}_l$ is defined in terms of sums whereas the sets N are described in terms of absolute values.

To prove Theorem 14 we use our previous Theorem 13 on the characterization of the Fourier transform $\hat{\mathfrak{J}}_l$. We want to produce a neighborhhod of zero N' in $\hat{\mathfrak{J}}_l$ such that for any $F \in N'$, we have $f \in N$.

For $j = 0, 1, 2, \cdots$, let $e_j = \max_{j' \geqq j} |a_j|$, so $\{e_j\}$ is montonically decreasing and $\sum e_j z^j$ is an entire function e of order l. We set $\gamma_1(x) = [1 + e(|x|)]^{-1}$.

We define $\gamma_2(y)$ as follows: $\gamma_2(0) = 1$. For each $j > 0$, let $j' = j'(j) > 0$ be so chosen that $\exp(jj') \geqq \max_{|x| \leqq j} b(x)$; then set $\gamma_2(\pm j') = \exp(jj')$. The definition of γ_2 is completed by requiring that γ_2 be continuous and monotonic.

Set

$$\gamma(z) = \gamma_1(x)\gamma_2(y)(1+x^2)^{-1}. \tag{46}$$

Let N' be the set of $F \in \hat{\mathfrak{J}}_l$ for which $|F(z)| \leqq \gamma(z)$ for all $z \in C$. By Theorem 13, N' is a neighborhood of zero in $\hat{\mathfrak{J}}_l$. We shall produce a $c > 0$ so that $F \in N'$ implies $cf \in N$.

Let $F \in N'$. Then for $t \geqq 0$,

$$\begin{aligned}
\Sigma |a_j f^{(j)}(t)| &= \Sigma |a_j \int x^j F(x) \exp(itx)\, dx| \\
&= \Sigma |a_j \int_{\mathcal{J}(z)=j'([t]+1)} z^j F(z) \exp(itz)\, dz| \\
&\leqq \Sigma |a_j| \int_{\mathcal{J}(z)=j'([t]+1)} |z|^j |F(z)| \exp(-tj'([t]+1))\, dz \\
&\leqq \exp(-tj'([t]+1)) \int_{\mathcal{J}(z)=j'([t]+1)} \Sigma |a_j| |z|^j |F(z)|\, dz \\
&\leqq \exp(-tj'([t]+1)) \int_{\mathcal{J}(z)=j'([t]+1)} e(|z|)\gamma(z)\, dz \\
&\leqq \exp(-tj'([t]+1)) \int e(|x|+j'([t]+1))\gamma_1(x)\gamma_2(j'([t]+1))(1+x^2)^{-1}\, dx \\
&\leqq \exp(-(t-1)j'([t]+1))\gamma_2(j'[t]+1) \int e(|x|)\gamma_1(x)(1+x^2)^{-1}\, dx
\end{aligned}$$

because we can assume, by modifying e if necessary that (see Lemma 8) $e(|x|+|\alpha|) \leqq e(|x|)\exp(|\alpha|)$, where we have written $[t]$ for the greatest integer $\leqq t$.

Now, we have, by definition,

$$\gamma_2(j'[t]+1) = \exp[([t]-2)j'([t]+1)]$$

so that

$$\begin{aligned}
\exp[-(t-1)j'([t]+1)]\gamma_2(j'[t+1]) &\leqq \exp(-j'([t]+1)) \\
&\leqq [\max_{|s| \leqq [t]+1} |b(s)|]^{-1} \\
&\leqq |b(t)|^{-1}.
\end{aligned}$$

Moreover, we have $\int e(|x|)\gamma_1(x)(1+x^2)^{-1} \leqq 1$. Thus,

$$\Sigma |a_j f^{(j)}(t)| \leqq |b(t)|^{-1} \tag{47}$$

if $t>1$. A similar argument applies if $t<-1$. Moreover, for any t we have

$$\begin{aligned}\sum |a_j f^{(j)}(t)| = \sum |a_j \int x^j F(x)\exp(itx)\,dx| &\leqq \int e(|x|)|F(x)|\,dx \\ &\leqq \int e(|x|)\gamma_1(x)(1+x^2)^{-1} \\ &\leqq 1.\end{aligned}$$

It follows that $[\max_{|t|\leqq 2}|b(t)|]^{-1} f \in N$, which is the desired result.

(2) The equivalence of (1) and (2) is readily verified.

Remark. It has been necessary to use the Fourier transform to prove Theorem 14. A direct proof of this theorem is undoubtedly very difficult to obtain.

Besides being useful in the study of differential equations and elliptic operators (see Ehrenpreis [7] and [10]), the spaces $\mathcal{J}_l$ exhibit a remarkable property as topological vector spaces: Let us denote, for any $t>0$, by $\mathcal{J}_l^t$ the space for all $f\in\mathcal{J}_l$ such that $f(x)=0$ if $|x|>t$. We give $\mathcal{J}_l^t$ the topology induced from $\mathcal{J}_l$. We call λ the topology of the inductive limit of the spaces $\mathcal{J}_l^t$ for t integral, so λ is another topology on $\mathcal{J}_l$ (see Dieudonné and Schwartz [1]. Strictly speaking, only the inductive limit of Frechet spaces is defined in Dieudonné and Schwartz [1], but the definitions do not require the Frechet property.) We shall show that the topology λ is the same as that of $\mathcal{J}_l$. Note that the analogous property does not hold for the space $\mathcal{D}$, namely, the topologices of $\mathcal{D}$ and $\mathcal{D}_F$ are different.

First we need

LEMMA 9. *For any $l<1$, we can find a sequence of non-negative functions $h_j\in\mathcal{J}_l$ such that*

a) carrier $h_j \subset [j'-1\leqq x\leqq j'+1]$ *for some j', and for at most 3 values of j does* carrier h_j *intersect* $[j'-1\leqq x\leqq j'+1]$.

b) *For any $x\in R$, $\sum h_j(x)=1$.*

Proof. Let

$$f_p(x) = \begin{cases}\exp(-x^{-p}) & \text{if } x>0\\ 0 & \text{if } x\leqq 0.\end{cases} \tag{48}$$

We want to obtain bounds on the derivatives of f_p. (The f_p are clearly indefinitely differentiable.) First we compute the maximum of $x^{-r}f_p(x)$. Differentiating we obtain, for $x>0$,

$$\begin{gathered}-rx^{-r-1}\exp(-x^{-p}) + px^{-r-p-1}\exp(-x^{-p}) = 0\\ p-rx^p=0, \qquad x=(p/r)^{1/p}.\end{gathered}$$

Thus the maximum of $x^r f_p(x)$ is $(r/p)^{r/p}\exp(-r/p)$.

Now, the k-th derivative of f_p is a sum of terms of the form $e_j x^{-j} f_p(x)$, where $j = l(p+1) + (k-l)$ for some $l \leqq k$, and then

$$e_j \leqq [l(p+1) + (k-l)]^{k-l}(p+1)^l.$$

Thus, using the above, we find that

$$\begin{aligned} \max e_j f_p(x) &\leqq (p+1)^l (lp+k)/p + kl \\ &\leqq (p+1)^l (lp+k)^{k(1+1/p)} \\ &\leqq (p+1)^l (kp+k)^{k(1+1/p)} \\ &\leqq (p+1)^{3k} k^{k(1+1/p)} \end{aligned}$$

because $l \leqq k$. Since the number of terms $e_j f_j(x)$ is 2^k, we see that

$$\max_{x \in R} |f_p^{(k)}(x)| \leqq 2^k (p+1)^{3k} k^{k(1+1/p)}. \tag{49}$$

Now we use (49); it follows that if $\sum a_j z^j$ is an entire function of order $\leqq l$, then $\sum a_j f_p^{(j)}(x)$ will converge uniformly for $x \in R$ if $p > l(1-l)$.

By forming products, translates, and integrals of the f_p, we can easily construct functions satisfying the hypotheses of Lemma 9.

Now we wish to describe the topology λ explicitly:

PROPOSITION 5. *Let $\{a^k(z)\} = \{\sum a_j^k z^j\}$ be a sequence of entire functions of order $\leqq l$. Let N be the set of $f \in \mathfrak{J}_l$ which satisfy*

$$\max_{k-1 \leqq |x| \leqq k+1} \sum |a_j^k f^{(j)}(x)| \leqq 1. \tag{50}$$

Then these sets N form a fundamental system of neighborhoods of zero for λ.

Proof. It follows immediately from Theorem 14 that $N \cap \mathfrak{J}_l^r$ is a neighborhood of zero in $\mathfrak{J}_l^r$ for all r; since N is convex, N is a neighborhood of zero for λ. Conversely, let M be a convex neighborhood of zero in $\mathfrak{J}_l^r$. We can assume that we can find an entire function of order $\leqq l$, $b^r(z) = \sum b_j^r z^j$, so that M^r contains the set of $f \in \mathfrak{J}_l^r$ with

$$\max_{x \in R} |\sum b_j^r f^{(j)}(x)| \leqq 1.$$

Let us note the following: For any $f \in \mathfrak{J}_l$, and any k, $f_k = h_k f$ (where the h_k are as in Lemma 9) is again in $\mathfrak{J}_l$. Moreover, we can write, for any $x \in R$,

$$\begin{aligned} \sum b_j^r f_k^{(j)}(x) &= \sum b_j^r (fh_k)^j(x) \\ &= \sum b_j^{r,k}(x) f^{(j)}(x), \end{aligned}$$

where for each x we have $\sum b_j^{r,k}(x) z^j$ is an entire function of order $\leqq l$ and,

moreover, if $c_j^{r,k} = \max_{x \in R} |b_j^{r,k}(x)|$, then $\sum c_j^{r,k} z^j$ is an entire function of order $\leqq l$. For each r only a finite number of k have the property that the carrier of k meets $[-r-1 \leqq x \leqq r+1]$; we define c_j^r as the maximum of $c_j^{r,k}$ over these k, so $\sum . c_j^r z^j$ is an entire function of order $\leqq l$.

We define N as the set of $f \in \mathfrak{J}_l$ for which

$$\sup_{r-1 \leqq |x| \leqq r+1} \sum |c_j^r f^{(j)}(x)| \leqq 3^{-r-3}. \tag{51}$$

We claim that $N \subset M$. For any $f \in N$ write $f = \sum f_k$, where $f_k = h_k f$. Let r be so chosen that *carrier* $h_k \subset [|x| \leqq r+1]$; by our construction there are at most $3r$ values of k corresponding to any value of r. Next we have, if $|x| \leqq r+1$,

$$\begin{aligned} |\sum b_j^r f_k^{(j)}(x)| &\leqq \sum |b_j^r f_k^{(j)}(x)| \\ &\leqq \sum |c_j^r f^j(x)| \\ &\leqq 3^{-r-3}. \end{aligned}$$

Thus, $3^{r+3} f_k \in M^r \subset M$. Since M is convex, we have $f = \sum_{\text{(finite)}} f_k \in M$. Thus $N \subset M$ which is the desired result.

THEOREM 15. *The topology λ is the same as that of $\mathfrak{J}_l$.*

Proof. Since the topologies induced by $\mathfrak{J}_l$ and λ on $\mathfrak{J}_l^r$ are, for any r, just the topology of J_l^r, it follows from the definition of an inductive limit (see Dieudonné and Schwartz [1]) that the topology λ is stronger than that of $\mathfrak{J}_l$. (This also follows immediately from Proposition 5.)

Let N be a neighborhood of zero for λ; by Proposition 5 we can find entire functions $a^k(z) = \sum a_j^k z^j$ of order $\leqq l$ so that N contains the set of $f \in \mathfrak{J}_l$ for which

$$\max_{k-1 \leqq |x| \leqq k+1} \sum |a_j^k f^{(j)}(x)| \leqq 1.$$

We define the sequence $\{b_n\}$ as follows: $b_0 = 1$, $b_n = \exp[-(l-l^2)\log n]$ for $n = 1, 2, \cdots, n_1$, where $n_1 \geqq 1$ is so chosen that $|a_j^k| \leqq \exp(-(l-l^2)\log j)$ whenever $k \leqq 2$ and $j \geqq n_1$. Suppose that b_n has been defined for $n \leqq n_p$, where $p \geqq 1$, and $n_p \geqq p$. Then we set

$$b_n = \exp(-(l-l^{p+2})) \quad \text{for } n = n_p + 1, n_p + 2, \cdots, n_{p+1},$$

where $n_{p+1} > n_p$ is so chosen that $|a_j^k| \leqq \exp[-(l-l^{p+3})]$ whenever $k \leqq p+2$ and $j \geqq n_{p+1}$. It is clear that the numbers n_p can be found and, moreover, that $\sum b_j z^j$ is an entire function of order l.

Now, we have, for any k, $|a_j^k| \leqq b_j$ for all but a finite number of j.

Thus, we can find a constant $c_k > 0$ with $c_k \mid a_j^k \mid \leqq b_j$ for all j. Call M the set of $f \in \mathcal{J}_1$ which satisfy, for all k,

$$\max_{k-1 \leqq |x| \leqq k+1} \sum \mid b_j f^{(j)}(x) \mid \leqq c_k. \tag{52}$$

By Theorem 14, M is a neighborhood of zero in $\mathcal{J}_1$; we claim that $M \subset N$.

Let $f \in M$, then if $k-1 < \mid x \mid \leqq k+1$,

$$\begin{aligned} \sum \mid a_j^k f^{(j)}(x) \mid &\leqq (1/c_k) \sum \mid b_j f^{(j)}(x) \mid \\ &\leqq 1 \end{aligned}$$

because of (52). Thus $f \in N$, which concludes the proof of Theorem 15.

Example 7'. We choose C_1 as the set of all sequences $\{a_n\}$ for which $a_n = O(\epsilon^n/\Gamma(rn))$ for some $\epsilon > 0$. L_1 consists of all sequences $\{b_n\}$ such that $\sum b_n z^n$ is an entire function. We define C_j and L_j to be trivial for $j > 1$. We denote the space $\mathcal{S}$ obtained by ${}_r\mathcal{D}$. The spaces ${}_r\mathcal{D}$ play an important role in the study of elliptic operators and Cauchy's problem in partial differential equation (see Ehrenpreis [7], [10]).

By using methods similar (and, in fact, simpler) than those employed in the proof of Theorems 11 and 13, we can deduce

THEOREM 11'. *${}_r\hat{\mathcal{D}}$ consists of all functions F of exponential type which satisfy for some $A > 0$,*

$$\mid F(x) \mid = O(\exp(-A \mid x \mid^{r'})), \tag{28'}$$

where $r' = 1/r$.

THEOREM 13'. *Let γ be any continuous positive function on C such that for all $A > 0$ and some k we have ($z = x + iy$)*

$$\exp(-A \mid x \mid^{r'} + k \mid y \mid) = O(\gamma(x, y)) \tag{29'}$$

and γ is the product of a function of x by a function of y. Let N be the set of $F \in {}_r\mathcal{D}$ which satisfy

$$\mid F(z) \mid \leqq \gamma(z) \text{ for all } z \in C.$$

Then these sets N form a fundamental system of neighborhoods of zero in ${}_r\hat{\mathcal{D}}'$.

II. Infinite derivatives for $\mathcal{E}$. By $\mathcal{E}$ we denote the space of all indefinitely differentiable functions on R with the Schwartz topology (see Schwartz [1]): $\mathcal{E}$ is metrizeable; a sequence $\{f^j\}$ converges to zero in $\mathcal{E}$ if and only if the f^j and all their derivatives converge to zero uniformly on every compact set of R.

Now, the space $\mathcal{S}$, which played a fundamental role in Chapter I, was defined by both regularity conditions and growth conditions at infinity. In contrast to this, the space $\mathcal{E}$ is defined solely in terms of regularity conditions. Thus, only the operator d/dx will be used to define the spaces of functions which we consider here, while only the operator X will be needed to define the space of Fourier transforms.

Let $a = \{a_j\}$ be a sequence of complex numbers. We say that $f \in \mathcal{E}$ is *in the domain of* D^a (*for the space* $\mathcal{E}$) if $\sum a_j f^{(j)}$ converges in the space $\mathcal{E}$ and if, moreover, for each $k > 0$ we have $\sum |a_j f^{(j+k)}(x)|$ converges uniformly for x in any compact set of R; when no confusion can arise, we shall merely state that f is in the domain of D^a, and we set $D^a f = \sum a_j f^{(j)}$.

Let A be a class of sequences $\{a_j\}$. We define the space $\mathcal{P}$ to consist of all $f \in \mathcal{E}$ which are in the domain of D^a for all $a \in A$. The topology of $\mathcal{P}$ is defined by the semi-norms

$$\sup_{x \in K} \sum |a_j f^{(j+k)}(x)| \tag{53}$$

for K any compact set in R and $\{a_j\}$ any sequence in A and any integer $k \geqq 0$

Remark. $\mathcal{P}$ may not be Hausdorf even if A is non-empty. E.g. A consists of the single sequence: $a_0 = 0$, $a_1 = 1$, $a_j = 0$ for $j > 0$. Then $f(x) = x$ is not separated from 0. The results of the theory of mean-periodic functions (see Schwartz [4], Ehrenpreis [5]) seem to indicate that if all the functions $\sum a_j z^j = a(z)$ for $\{a_j\} \in A$ are entire functions, then $\mathcal{P}$ will be Hausdorf if, for any $\{a_j\} \in A$, the sequence $\{a_j'\}$ also lies in A where $\{a_j'\}$ is obtained from $\{a_j\}$ by some small deformation. $\mathcal{P}$ will not be Hausdorf if the $a(z)$ have a common zero z_0, for then $\exp(z_0 x)$ will satisfy $D^a \exp(z_0 x) = 0$ for all $a \in A$, and thus cannot be separated from 0.

Problem. Can we replace the semi-norms in (53) by the semi-norms

$$\sup_{x \in K} |\sum a_j f^{(j+k)}(x)| = \sup_{x \in K} |D^a f^{(k)}(x)| \,? \tag{54}$$

This problem seems to be very difficult. In particular, I do not know whether or not the semi-norms (54) lead to a complete space.

The set A will be called *admissible* if every finite sequence belongs to A. It is readily verified that the corresponding space $\mathcal{P}$ is always Hausdorf if A is admissible. We shall assume in the following that any set of sequences A that we consider is admissible. A will be called trivial if it consists exactly of all finite sequences; in this case $\mathcal{P} = \mathcal{E}$.

PROPOSITION 6. *$\mathcal{P}$ is a complete locally convex Hausdorf topological vector space.*

Proof. Everything is obvious except the completeness of $\mathcal{P}$. Let ϕ be a Cauchy filter base in $\mathcal{P}$; since the topology of $\mathcal{P}$ is stronger than that of $\mathcal{E}$, ϕ converges to some element $f \in \mathcal{E}$. We claim that $f \in \mathcal{P}$ and ϕ converges to f in the topology of P.

We know that for any compact set $K \subset R$, for $f_\alpha \in \phi$, $f_\alpha \to f$, we have $\sum |a_n(d^n f_\alpha/dx^n)(x)|$ converges uniformly for all α and for $x \in K$; we claim it converges to $\sum |a_n(d^n f/dx^n)(x)|$. If we write $b_\alpha{}^n(x) = a_n f_\alpha{}^{(n)}(x)$, and $b^n(x) = a_n f^{(n)}(x)$, then the above is the statement that the conditions $b_\alpha{}^n(x) \to b^n(x)$ uniformly for $x \in K$, $\sum |b_\alpha{}^n(x)|$ converges uniformly for all α and for $x \in K$, imply $\sum |b^n(x)| < \infty$ and $\sum |b_\alpha{}^n(x)|$ converges to $\sum |b^n(x)|$ uniformly for $x \in K$. This is seen as follows:

Given any n_0 however large, we can find a cofinal set of α for which $|b_\alpha{}^n(x) - b^n(x)| \leqq 1/n_0$ for $n \leqq n_0$ and all $x \in K$. Thus,

$$\begin{aligned}\sum_{n \leqq n_0} |b^n(x)| &\leqq \sum_{n \leqq n_0} |b_\alpha{}^n(x)| + \sum_{n \leqq n_0} |b_\alpha{}^n(x) - b^n(x)| \\ &\leqq 1 + \sum_n |b_\alpha{}^n(x)| \\ &\leqq \text{const.},\end{aligned}$$

where the constant is independent of n_0.

Finally, it is immediate that $\sum |a_n((d^n f_\alpha/dx^n)(x) - (d^n f/dx^n(x))| \to 0$. A similar argument works for the derivatives of f and gives our result.

In dealing with the Fourier transform, it is best to consider the Fourier transform of $\mathcal{P}'$ which is the dual of $\mathcal{P}$ with the topology of uniform convergence on the bounded sets of $\mathcal{P}$. For the Fourier transform of $\mathcal{P}'$ is, in general, a space of entire functions which can be described explicitly in many cases (compare Ehrenpreis [1], [3], and [8]). In fact, in many cases we get again a space which can be described by infinite derivatives.

We shall now consider several examples of spaces $\mathcal{P}$.

Hypothesis I. (α). The sequence $\{a_j\}$ belongs to A if $a(z) = \sum a_j z^j$ is an entire function of exponential type.

(β) If $\{a_j\} \in A$ and $c_j = O(1 + p^j)$ for some p, then $\{\sum_{i=m}^{n} c_j a_{j-i} l^i / i!\}$ is in A for any l whenever $0 \leqq m \leqq n \leqq j$.

(γ). For any $f \in \mathcal{P}$, the series $\sum f^{(j)} X^j / j!$ converges to f in the topology of $\mathcal{P}$.

From Hypothesis I (α) it follows easily that all the functions in P are entire functions. Moreover, the topology of $\mathcal{P}$ is stronger than the topology of the space $\mathcal{H}$ of entire functions on the complex plane (see Ehrenpreis [5]).

Now, for any $S \in \mathcal{P}'$, we define the Fourier transform $T = \mathcal{F}(S)$ as the formal power series $T(z) = \sum T_j z^j$, where $T_j = S \cdot i^j X^j / j!$. In most cases T will be an analytic function. We denote by $\hat{\mathcal{P}}'$ the space of Fourier transforms of $\mathcal{P}'$ with the topology to make $\mathcal{F}$ a topological isomorphism.

For any set A of sequences we denote by A^{-1} the set of all sequences $\{b_j\}$ such that $\sum |b_j a_j| < \infty$ for all $\{a_j\} \in A$. It is clear that A^{-1} is always admissible.

THEOREM 16. *Let A satisfy Hypothesis* I. *Then the space $\mathcal{P}'$ is the space of infinitely differentiable functions defined by the operators D^b for $b \in A^{-1}$.*

Proof. First we note that, for any $b = \{b_j\} \in A^{-1}$, we have $\{b_j(1+j^2)X^j\}$ is bounded in the topology of $\mathcal{P}$. For, given any $\{a_j\} \in A$ and any interval $[-l \leqq x \leqq l]$ in R, $(l \geqq 1)$, we have

$$\sup_{|x| \leqq l} |(1+j^2) a_j b_j | x |^j| = (1+j^2) |a_j b_j| l^j.$$

Now, by Hypothesis I(β), $\{a'_j\} = \{\sum_{i=0}^{j} (1+j^2) |a_{j-i}| l^i / i!\}$ is again in A, so that

$$\begin{aligned} \sup_{|x| \leqq l} |\{D^a[(1+j^2) b_j X^j)]\}(x)| &\leqq \sum_{i=0}^{j} (1+j^2) |b_j a_{j-i} j! l^i| / i! \\ &= j! |b_j| (i+j^2) \sum_{i=0}^{j} |a_{j-i}| l^i / i! \\ &= |b_j a'_j| j!. \end{aligned}$$

Now, by hypothesis, $|b_j a'_j|$ is bounded in j. Thus,

$$\sup_{|x| \leqq l} |[D^a(b_j X^j / j!)](x)| \leqq \text{const.}/(1+j^2)$$

which is the desired result.

It follows that, for any $S \in \mathcal{P}'$, $\{S \cdot (1+j^2) b_j i^j X^j / j!\} = \{(1+j^2) b_j T_j\}$ is a bounded set of complex numbers. Thus, $\sum |b_j T_j| < \infty$. It is also clear from the above that $\sum |b_j T_j|$ is uniformly bounded for T in any bounded set of $\hat{\mathcal{P}}'$.

Conversely, suppose that T has the property that $\sum |b_j T_j| < \infty$ for any $\{b_j\} \in A^{-1}$; we want to show that $T \in \mathcal{P}'$. For any $f \in \mathcal{P}$, write $f(z) = \sum f_n z^n$, where, by hypothesis, the series converges to f in the topology of E. Then we define S by $S \cdot f = \sum f_n T_n (-i)^n n!$. We must show, of course, that this series converges. But we know that $\sum |a_n f_n n!|$ converges for any $\{a_n\} \in A$, by the definition of $\mathcal{P}$.

LEMMA 10. $(A^{-1})^{-1} \subseteq A$ *for any admissible A.*

Lemma 10 is an immediate consequence of the definitions.

From Lemma 10, it follows that $\{(-i)_n T_n\} \in A$, and this implies that $f \to \sum f_n T_n (-i)n!$ defines a continuous linear function L on $\mathfrak{P}$. It is clear that the Fourier transform of L is just T, and this completes the proof of Theorem 16.

We have still to prove the statement in the theorem about the topology of $\hat{\mathfrak{P}}'$, but this can be accomplished by the above methods.

We want to use our theory of infinite derivatives to describe the topology of functions which are analytic on an arbitrary set B in the complex plane. The method is easily extended to sets in complex affine space of n dimensions. Now, if B is open, the compact-open topology is obviously the most natural one. For a more general class of B, a natural topology has been constructed by Van Hove (see Van Hove [1], Waelbroeck [1], Grothendieck [1], [2], Köthe [2]). This is defined as follows: Let $B_1, \cdots, B_r, \cdots$ be a sequence of open neighborhoods of B with $B_{j+1} \subset B_j$, $\cap B_j = B$ and each B consists of only a finite number of connected components each of which meets B. Let $\mathfrak{H}(B_j)$ denote the space of functions which are analytic on B_j with the compact-open topology. Then we define $\mathfrak{H}(B)$ as the inductive limit of the spaces $H(B_j)$ (see Köthe [1]), that is, $f \in \mathfrak{H}(B)$ means that f is analytic on some neighborhood of B. $\mathfrak{H}(B)$ is given the strongest locally convex topology which makes the natural maps $\mathfrak{H}(B_j) \to \mathfrak{H}(B)$ continuous.

Our definition of the topology by means of infinite derivatives is superior to the Van Hove topology because it is intrinsic, that is, it is defined in terms of the set alone without using the fact that it is a subset of the complex plane. Because of this, we are able to use our methods to define the topology on the real analytic functions on a real analytic manifold (with singularities). In addition, our method leads to the definition of the Fourier transform on these spaces. We hope to discuss these notions in detail in a future publication.

We shall change our previous notations slightly for convenience: We denote by $\mathcal{E}$ the space of indefinitely differentiable functions in the complex $z = x + iy$ plane with the Schwartz topology (Schwartz [1]). For B any subset of the plane, let $\bar{B}$ denote the closure of B. We also define $A = \{a_n\}$ such that

$$a_n = O(\epsilon^n/n!) \text{ for any } \epsilon > 0. \tag{54}$$

Definition. Two functions which are analytic on a neighborhood of B are called equivalent if they coincide on a neighborhood of B. An equivalence class of functions analytic on neighborhoods of B is called an analytic function on B.

We have the obvious

LEMMA 11. *Let f_1 and f_2 be equivalent functions which are analytic on B. Then for any n and any $z_0 \in B$,*

$$(d^n f_1/dz^n)(z_0) = (d^n f_2/dz^n)(z_0).$$

Thus we can make the following

Definition. $\mathfrak{H}(B)$ is the space of analytic functions on B. A fundamental system of neighborhoods N of zero in $\mathfrak{H}(B)$ is defined as follows: Let K be a compact subset of B and let $\{a_n\} \in A$. Then N consists of all $f \in \mathfrak{H}(B)$ which satisfy

$$\max_{z \in K} \sum |a_n f^{(n)}(z)| \leqq 1. \tag{55}$$

PROPOSITION 7. *The following three topologies on the set of functions of $\mathfrak{H}(B)$ are equivalent:*

1. *The topology of $\mathfrak{H}(B)$.*

2. *The topology defined by the semi-norms*

$$\max_{n,z} a_n |f^{(n)}(z)|$$

for $\{a_n\} \in A$.

3. *The topology defined by the semi-norms*

$$\sum a_n \max_z |f^{(n)}(z)|$$

for $\{a_n\} \in A$.

Proof. It is clear that of the three topologies, 3 is the strongest and 2 is the weakest; thus we have to show that 2 is stronger than 3. Now, if $\{a_n\} \in A$, then so is $\{a_n n^2\}$. Thus, if $b > 0$ is properly chosen, any $f \in H(B)$ which satisfies $\max_z (a_n n^2) |f^{(n)}(z)| \leqq 1$ must satisfy $\sum a_n \max_z |f^{(n)}(z)| \leqq b$. Thus Proposition 7 is proven.

PROPOSITION 8. *Assume B possesses the following topological properties:*

1. *Every connected component of B is open in B.*

2. *Every connected component of B is analytically arcwise connected, that is, any two points belonging to the same component can be joined by a real analytic curve in B.*

3. *For each $z \in B$ let $U(z)$ be an open set in the complex plane. Then we can find a collection of sets $\{U_\alpha\}$ which are open in the complex plane*

and cover B such that each U_α is contained in some $U(z)$ and, moreover, if $U_\alpha \cap U_{\alpha'} \neq \phi$, $B \cap U_\alpha \cap U_{\alpha'} \neq \phi$.

Then $\mathfrak{A}(B)$ is a complete, Hausdorf, locally convex topological algebra. If B is open, then the topology of $\mathfrak{A}(B)$ is the compact-open topology.

Remark. Hypothesis 2 can be weakened to: Any two points in the same component can be joined by a piecewise analytic curve in B, and the proof is essentially the same as that given below. However, I do not know of Proposition 8 is true for an arbitrary set B.

Proof. It is clear that $\mathfrak{A}(B)$ is a locally convex Hausdorf topological vector space. To show the completeness of $\mathfrak{A}(B)$ we need the following

LEMMA 12. *Let $f^\alpha(z) = \sum (f^\alpha{}_n/n!)z^n$ be a formal power series with the property that $\sum | f^\alpha{}_n a_n |$ are uniformly bounded in α whenever $a_n = O(\epsilon^n/n!)$ for any $\epsilon > 0$. Then all $f^\alpha(z)$ are analytic in a fixed neighborhood of $z = 0$.*

Proof of Lemma. Assume that f^α is not analytic in a fixed neighborhood of $z = 0$. Then we can find an infinite sequence of positive numbers $n_j \to \infty$ and a sequence $\{\alpha_j\}$ such that

$$| f^{\alpha_j}{}_{n_j} | \geqq n_j!\, j^{n_j}.$$

Now, let

$$a_n = \begin{cases} j^{-n}/n! & \text{when } n = n_j \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $a_n = O(\epsilon^n/n!)$ for any $\epsilon > 0$, but

$$\sum | a_{n_j} f^{\alpha_j}{}_{n_j} | = \sum | f^{\alpha_j}{}_{n_j} |\, j^{-n_j}/n_j!$$
$$\geqq \sum 1 = \infty.$$

This concludes the proof of the lemma.

Proof of Proposition continued: Let $\{f^\alpha\}$ be a Cauchy filter base in $\mathfrak{A}(B)$; then f^α clearly converges to a function f which is continuous on B and is analytic at each interior point of B. For each n we define the continuous function $f_n(z) = \lim (d^n/dz^n) f^\alpha(z)$. For each $z_0 \in B$ we claim that the formal power series $\sum f_n(z_0)(z - z_0)^n$ converges in the neighborhood of z_0. For simplicity of notation we assume that $z_0 = 0$. Now, reasoning as in the proof of Proposition 6 we deduce that $\sum | f_n(0) a_n | < \infty$ whenever $\{a_n\} \in A$. By Lemma 12 we see that $\sum f_n(0) z^n$ is analytic in the neighborhood of $z = 0$.

Now, we have difficulty in constructing the limit of f^α because $N(z_0)$ and $N(z_1)$ may overlap without $\sum f_n(z_0)(z - z_0)^n$ and $\sum f_n(z_1)(z - z_1)^n$ being

equal in the intersection. However, by a simple argument we can show that we can choose neighborhoods $N'(z) \subset N(z)$ for any $z \in B$ with the additional property that $N'(z_0) \cap N'(z_1)$ is empty if z_0 and z_1 do not belong to the same connected component of B; namely, we just make sure that $N'(z)$ is contained in the circle, center z, radius less tha none-third the distance from z to the complement in B of the connected component of B containing z. The positivity of this distance is a result of the fact that z cannot be a limit point of points of B which do not lie in the same conected component, since the components are assumed to be open. By our assumptions on B we may also assume that $N'(z)$ have the property that if $N'(z) \cap N'(z') \neq \phi$ then $B \cap N'(z) \cap N'(z') \neq \phi$. (Of course, $N'(z)$ are open in the complex plane and cover B but $N'(z)$ are defined only for some z.)

Call $N = \bigcup_{z \in B} N'(z)$ and define g on N by $g(z) = \sum f_n(z_0)(z - z_0)^n/n!$ for $z \in N(z_0)$. We have three things to show: a. $g = f$ on B. b. g is well-defined. c. g is analytic on N.

Proof of a. *Let* $z_1 \in B$, then z belongs to some $N'(z_0)$. If $z_1 = z_0$ then certainly $f(z_1) = g(z_1)$. If $z_1 \neq z_0$ then by construction there exists a real analytic curve Γ lying in B and joining z_1 and z_0. It is a simple matter using integration on Γ to show that f is an indefinitely differentiable function on Γ and for all n we have $f_n(z) = (d^n/dz^n)f(z)$ for $z \in \Gamma$, where $d^n/dz^n f$ is defined by using difference quotients on Γ only; that is,

$$(df/dz)(z_2) = \lim_{z \in \Gamma,\ |z - z_2| \to 0} (f(z_2) - f(z))/(z_2 - z).$$

Moreover, since Γ is compact, the definition of our topology shows that for any $\{a_n\} \in A$ we must have $\sum |a_n f_n(z)|$ uniformly bounded for $z \in \Gamma$. Lemma 12 then tells us that there is a $\delta > 0$ so that all the power series $\sum f_n(z)(z' - z)^n/n!$ converge uniformly for $|z' - z| \leqq \delta$ and for $z \in \Gamma$. We now apply Pringsheim's theorem (see Mandelbrojt [2]): This tells us that f is real analytic on Γ. (Strictly speaking, Pringsheim's theorem is proven only for Γ a segment of the real axis, but there is no difficulty in extending the proof to any real analytic curve.)

Now, it is clear that g and f both have all their derivatives equal at z_0. Thus, since g and f are real analytic on Γ they are equal for all points of Γ. Thus, $g(z_1) = f(z_1)$ which is the desired result.

Proof of b. Let z_0, z_0' be points in B such that $N'(z_0) \cap N'(z_0')$ is not empty. Then by construction, $B \cap N'(z_0) \cap N'(z_0') \neq \phi$, so let $z_1 \in B \cap N'(z_0) \cap N'(z_0')$. Then we can join z_1 to z_0 and z_1 with analytic arcs

Γ_1, Γ_2 respectively. By a. above, $C f_n(z_0)(z-z_0)^n/n!$ is equal to f on $(\Gamma_1 \cup \Gamma_2) \cap N'(z_0) \cap N'(z_0')$ and so is also equal to $\sum f_n(z_0')(z-z_0)^n/n!$. Since $\Gamma_1 \cup \Gamma_2$ is an arc and $N'(z_0) \cap N'(z_0')$ is open, the two functions

$$\sum f_n(z_0')(z-z_0')^n/n! \text{ and } \sum f_n(z_0)(z-z_0)^n/n!$$

must be equal on $N'(z_0) \cap N'(z_0')$. This proves that g is well defined.

Proof of c. This is clear from the definition.

It remains to show that f^α converges to f in the topology of $\mathfrak{A}(B)$ but this is clear. Thus, $\mathfrak{A}(B)$ is complete.

We show next that $\mathfrak{A}(B)$ is a topological algebra. This follows without difficulty from Leibnitz's formula for the derivative of a product:

$$(d^n/dx^n)(fh) = \sum_{j \leqq n} C_j^n f^{(j)} h^{(n-j)}.$$

Thus, if $a_n \geqq 0$,

$$\begin{aligned} \sum_n a_n |(d^n/dx^n)(fh)| &\leqq \sum_n \sum_{j \leqq n} a_n C_j^n |f^{(j)}| |h^{(n-j)}| \\ &= \sum_j \sum_k a_{j+k} C_j^{j+k} |f^{(j)}| |h^{(k)}|. \end{aligned} \tag{56}$$

Let us note that for $b_j \geqq 0$,

$$(\sum b_j |f^{(j)}|)(\sum b_k |h^{(k)}|) = \sum_{j,k} b_j b_k |f^{(j)}| |h^{(k)}|.$$

Thus we can make the left side of (56) small if we make $a_{j+k} C_j^{j+k} \leqq b_j b_k$ for appropriate $\{b_j\}$.

Now, $a_{j+k} = O(\epsilon^{j+k}/(j+k)!)$ for any $\epsilon > 0$. Thus

$$\begin{aligned} a_{j+k} C_j^{j+k} &= O(\epsilon^{j+k}(j+k)!/(j+k)!j!k!) \\ &= O[(\epsilon^j/j!)(\epsilon^k/k!)] \end{aligned} \tag{57}$$

for any $\epsilon > 0$.

Using (57) we define $\{b_j\}$ as follows: Let $L = \max(j! a_j, 1)$. For each $l \geqq 1$, let p_l be chosen so large that $a_j \leqq K(1/l)^j (1/j!)$ for $j \geqq p_l$. Define $b_j = L^l (1/l)^j (1/j!)$ for $p_l \leqq j < p_{l+1}$. Then (56) and (57) show that the conditions

$$\max_{x \in K} \sum |b_j f^{(j)}(x)| \leqq 1, \qquad \max_{x \in K} \sum |b_j h^{(j)}(x)| \leqq 1$$

imply

$$\max_{x \in K} \sum |a_j (fh)^{(j)}(x)| \leqq 1$$

which proves that $\mathfrak{A}(B)$ is a topological algebra.

If B is open, then the topology of $\mathfrak{A}(B)$ is the compact-open topology

as is readily verified by means of Cauchy's integral. This completes the proof of Proposition 7.

In general, we do not know if the topology of $\mathcal{H}(B)$ is the same as the Van Hove topology; we shall give some examples later in which the two topologies are the same.

PROPOSITION 9. *Let h be a complex analytic mapping of B into B_1, that is, $h \in H(B)$ and $h(B) \subset B_1$. Define $h^*: H(B_1) \to H(B)$ by $h^*(f)(x) = f(h(x))$. Then h^* is a continuous linear map.*

Proof. h^* is clearly linear so we need verify at zero only. Let K be a compact set in B; since h is continuous, $h(K)$ is compact. Let $\{a_j\}$ be a sequence satisfying (54); let N be the set of $g \in H(B)$ which satisfy

$$\max_{x \in K} \sum a_j \, | \, g^{(j)}(x) \, | \leqq 1. \tag{58}$$

Then we want to find a sequence $\{b_j\}$ satisfying (54) such that $f \in H(B_1)$,

$$\max_{x \in h(K)} \sum b_j \, | \, f^{(j)}(x) \, | \leqq 1 \tag{59}$$

should imply $h^* f \in N$.

The sequence $\{b_j\}$ can be constructed by a method which is essentially the same as the classical proofs by power series methods that an analytic function of an analytic function is analytic. We shall omit the details.

We wish to give some examples:

Example 1. *Functions analytic at the origin.* We denote the space of functions analytic at the origin by $\mathcal{B}$. We show that the topology of $\mathcal{B}$ is the same as that of Van Hove. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n = O(\epsilon^n/n!)$ for any $\epsilon > 0$, and let N be the set of $f \in \mathcal{B}$ satisfying $\sum | \, a_n f^{(n)}(0) \, | \leqq 1$. Suppose there exists a sequence $\{n_q\}$ with $n_0 = 0$ so that

$$a_n = 2^q q^{-n}/n! b_q \text{ for } n_{q-1} \leqq n < n_q, \, q \geqq 1. \tag{58}$$

We can write

$$f(x) = \sum_{q=1}^{\infty} 2^q f_q(x), \tag{59}$$

where $f_q(x) = 2^q \sum_{n_{q-1}}^{n_q - 1} f_n x^n$; the series on the right clearly converges in a neighborhood of zero. Moreover, we have for $| \, x \, | \leqq q^{-1}$,

$$
\begin{aligned}
|f_q(x)| &\leqq 2^q \sum_{n=n_{q-1}}^{n_q-1} |f_n| q^{-n} \\
&= b_q \sum_{n=n_{q-1}}^{n_q-1} |f_n| a_n n! \\
&\leqq b_q \sum_{n=0}^{\infty} |f_n| a_n n! \\
&= b_q \sum_{n=0}^{\infty} |f^{(n)}(0)| a_n \\
&\leqq b_q.
\end{aligned}
$$

Thus, $|f_q(x)| \leqq b_q$ for $|x| \leqq q^{-1}$.

Now, suppose we start with a closed convex neighborhood N of zero in $\mathfrak{B}$ in the Van Hove topology. Then for each $q > 0$ there is a $b_q > 0$ so that $\mathfrak{B}$ contains the set of f which are analytic in $|x| \leqq q^{-1}$ and satisfy $|f(x)| \leqq b_q$ for $|x| \leqq q^{-1}$. We can clearly choose a sequence $\{n_q\}$ so that if we define a_n by (58), then $a_n = O(\epsilon^n/n!)$ for any $\epsilon > 0$. (It suffices to take $n_{q-1} > 2^q b_q^{-1}$ for all $q > 1$.) Thus the neighborhood M of zero in $\mathfrak{B}$ defined by $\sum a_n |f^{(n)}(0)| \leqq 1$ has, by the above, the following property: each $f \in M$ can be written in the form $\sum_{q=1}^{\infty} 2^{-q} f_q$, where $f_q \in \mathfrak{B}$. It is readily verified that the series $\sum 2^{-q} f_q$ converges in the topology of Van Hove. Since $\mathfrak{B}$ is closed and convex by construction, this means that $f \in N$ which proves Proposition 9. The topology of $\mathfrak{B}$ is the same as the Van Hove topology.

Example 2. *Functions analytic in the closed unit disc.* Let us denote by K the unit disc, that is, the set of x with $|x| \leqq 1$.

THEOREM 17. *The following three topologies on $\mathfrak{A}(K)$ are equivalent:*

(1) *The topology of $\mathfrak{A}(K)$.*

(2) *The Van Hove topology on K.*

(3) *The topology defined by the semi-norms $\sum b_n |f^{(n)}(0)|$, where $\{b_j\}$ is a sequence of positive numbers satisfying $b_n = O((1+\epsilon)^n/n!)$ for any $\epsilon > 0$.*

Proof. Exactly as in the proof of the previous Proposition 9 we can show that the topology described in (3) is stronger than the Van Hove topology.

If U is any open set containing K and if β is bounded in $\mathfrak{A}(U)$, then β is clearly bounded in the topology (3). Thus, by general properties of the Van Hove topology, the toplogoy (2) is stronger than (3); hence, the topologies (2) and (3) are the same.

On the other hand, the Van Hove topology is stronger than the topology of $\mathcal{H}(K)$; Theorem 7 will thus be proven if we can show that topology (1) is stronger than (3). Now, for any function $f \in \mathcal{H}(K)$, f defines a function in L_2 of the unit circle. We have clearly (normalizing the measure so the unit circle gets measure 1)

$$[\int |f(e^{i\theta})|^2 d\theta]^{\frac{1}{2}} \leqq \max_{|x|\leqq 1} |f(x)|. \tag{60}$$

By Parseval's theorem the left side of (60) is $\sum |f_n|^2$. Thus we see that

$$\max_{|x|\leqq 1} |f(x)| \geqq [\sum |f_n|^2]^{\frac{1}{2}}. \tag{61}$$

By Schwart's inequality, $\sum |f_n|/(1+n^2) \leqq \text{const}\,[\sum |f_n|^2]^{\frac{1}{2}}$. Combining this with (61) we deduce

$$\max_{|x|\leqq 1} |f(z)| \geqq \text{const} \sum |f_n|/(1+n^2). \tag{62}$$

Now, let $\{a_n\}$ be a given sequence satisfying $a_n = O(\epsilon^n/n!)$ for all $\epsilon > 0$. We consider the set of f for which $\sum_{x \in K} a_j \max |f^{(j)}(x)| \leqq 1$. For each j we have by (62)

$$\max_{|x|\leqq 1} |f^{(j)}(x)| \geqq \text{const} \sum_{n \geqq j} |f_n| n!/(1+n^2)(n-j)!.$$

Thus,

$$1 \geqq \sum_j a_j \sum_{n \geqq j} |f_n| n!/(1+n^2)(n-j)!$$
$$= \sum c_n |f_n| n! = \sum c_n |f^{(n)}(0)|$$

say, where

$$c_n = (1+n^2)^{-1} \sum_{j \leqq n} a_j/(n-j)!. \tag{63}$$

The binomial expansion $(1+\alpha)^n = \sum_{j \leqq n} \alpha^j n!/j!(n-j)!$ shows that

$$c_n = O((1+\epsilon)^n/n!) \text{ for all } \epsilon > 0.$$

We are not finished, however! Given a sequence $\{b_n\}$ satisfying $b_n = O((1+\epsilon)^n/n!)$ for all $\epsilon > 0$ we have to produce a sequence $\{a_n\}$ satisfying $a_n = O(1+\epsilon)^n/n!)$ for all $\epsilon > 0$ so that for the corresponding sequence $\{c_n\}$ we have $b_n \leqq c_n$. The construction of $\{a_n\}$ is as follows: For each $q \geqq 1$ we define an integer d_q: $d_q \geqq 1$ is defined so that for $n \geqq d_q$ we have $b_n \leqq (1+1/q)^n/n!(1+n^2)$. d_0 is defined to be zero. We may clearly assume that $b_n \leqq 1/n!(1+n^2)$ for all n. Then we define

$$a_n = q^{-n}/n! \text{ for } d_{q-1} \leqq n < d_q. \tag{64}$$

For $n \geqq d_q$, we have by (64), $a_n \geqq q^{-n}$. Thus, for $n \geqq d_q$,

$$\begin{aligned} c_n &= n!(1+n^2)^{-1} \sum_{j \leqq n} a^j/(n-j)! \\ &\geqq n!(1+n^2)^{-1} \sum_{j \leqq n} q^{-j}/(n-j)!j! \\ &= (1+n^2)^{-1}(1+1/q)^{-n} \\ &\geqq b_n. \end{aligned}$$

This completes the proof of Theorem 17.

Example 3. *Functions analytic on the unit circle.* We denote by $\dot{K}$ the unit circle, i.e. the boundary of K. Then we have

THEOREM 18. *The following topologies on $\mathfrak{A}(\dot{K})$ are equivalent:*

(1) *The topology of $\mathfrak{A}(\dot{K})$.*

(2) *The Van Hove topology on K.*

(3) *The topology defined by the semi-norms*

$$\sum b_n |f_n|, \text{ where } f_n = \int_0^{2\pi} f(\theta) e^{-in\theta}\, d\theta$$

and where $\{b_n\}$ is a sequence of positive numbers satisfying $b_n = O(1+\epsilon)^n$ for any $\epsilon > 0$.

The proof is similar to the proof of the previous theorem and so will be omitted.

Similar results apply to the annulii

$$\alpha < |x| \leqq 1, \quad \alpha \leqq |x| \leqq 1, \quad \alpha \leqq |x| < 1$$

for any α with $0 \leqq \alpha < 1$. Now, by Proposition 9 our topology is invariant under conformal mappings which are real analytic on that part of the boundary of K which is in K; the same is true of the Van Hove topology (see Waelbruck [1]). Thus, the topology of $\mathfrak{A}(B)$ is the same as the Van Hove topology for B in case B is obtained from a domain of genus zero with real analytic boundary by adding certain parts of the boundary.

We do not know of any similar results for domains of genus > 1.

We conclude this section with a different application of the theory of infinite derivatives. This application is of great importance in obtaining very deep results on Cauchy's problem (see Ehrenpreis [10]).

For each r with $1 \leqq r < \infty$ we define ${}_r\mathcal{E}$ as the space of all $f \in \mathcal{E}$ for which the series $\sum a_n f^{(n)}$ converges in the space $\mathcal{E}$ whenever

$$a_n = O(\epsilon^n/\Gamma(rm)) \text{ for some } \epsilon > 0. \tag{65}$$

The topology of ${}_r\mathcal{E}$ is defined by means of the semi-norms $\sum a_n \max_{x \in K} |f^{(n)}(x)|$ for K a compact set in R and $\{a_n\}$ as above. The spaces ${}_r\mathcal{E}$ are easily shown to be complete topological vector spaces which are Schwartz spaces (hence reflexive). ${}_1\mathcal{E}$ is the space of entire functions and we shall write $\mathcal{E} = {}_\infty\mathcal{E}$. It is readily verified that ${}_r\mathcal{E}$ is metrizable. For $r \neq 1$ the spaces ${}_r\mathcal{E}$ bear the same relation to ${}_r\mathcal{D}$ that the space $\mathcal{E}$ does to the space $\mathcal{D}$.

${}_r\mathcal{E}$ is the dual of $\mathcal{E}$, ${}_r\hat{\mathcal{E}}'$ is the Fourier transform of ${}_r\mathcal{E}'$, as described above. For any $S \in {}_r\mathcal{E}'$, the Fourier transform $\mathfrak{F}(S)$ is the function

$$\mathfrak{F}(S)(z) = S \cdot \exp(iz\,\cdot)$$

for $z \in C$. It is easily seen that $\mathfrak{F}(S)$ is an entire function of exponential type.

THEOREM 19. *${}_r\hat{\mathcal{E}}'$ consists of all entire functions F of exponential type which satisfy, for some A,*

$$|F(z)| = O(|P(z)|)\exp[A\,|Rz|^{r'} + |I(z)|], \tag{66}$$

where P is any polynomial, and where r' is defined by

$$r' = 1/r \text{ for } r \neq \infty, \quad \infty' = 0. \tag{67}$$

The topology of ${}_r\hat{\mathcal{E}}'$ is defined as follows: Let $H(z)$ be a positive continuous function on C such that all $B > 0$

$$\exp[B\,|Rz|^{r'} + |I(z)|] = O(H(z)). \tag{68}$$

Let N_H be the set of $F \in {}_r\hat{E}'$ for which

$$|F(z)| \leqq H(z)$$

for all $z \in C$. Then the sets N_H form a fundamental system of neighborhoods of zero in ${}_r\hat{\mathcal{E}}'$.

Proof. Let $S \in \mathcal{E}'$. Now, if a_n are positive numbers satisfying $a_n = O(\epsilon^n/\Gamma(rn))$ for some $\epsilon > 0$, then we have

$$\begin{aligned} \sum a_n\,|(d^n/dx^n)\exp(iz\cdot x)| &= |\exp(iz\cdot x)|\sum a_n\,|(iz)^n| \\ &\leqq \exp(|x|\,|\mathfrak{I}(z)|)\sum a_n\,|z|^n. \end{aligned} \tag{69}$$

Now,

$$\begin{aligned}\sum a_n \mid z \mid^n &\leqq \text{const} \sum \epsilon^n \mid z \mid^n/\Gamma(rn+1)\\ &= \text{const} \sum (\epsilon \mid z \mid)^n/\Gamma(rn+1)\\ &= Q(\epsilon \mid z \mid),\end{aligned}$$

where

$$Q(z) = \sum z^n/\Gamma(rn+1).$$

For $x \geqq 0$, $Q(x)$ behaves like $\exp(x^r)$. This can be seen as follows:

$$\begin{aligned}\exp(x^r) &= \sum_{n=0}^{\infty} x^{r'n}/n!\\ &= \sum_{rm=0}^{\infty} x^m/\Gamma(rm+1).\end{aligned}$$

LEMMA 13. *For $x \geqq 1$ we have*

$$\alpha Q(x) \leqq \exp(x^r) \leqq \beta Q(2x) \tag{70}$$

for some constants α, β which depend only on r.

Proof of Lemma. We write first

$$\exp(x^r) = \sum_{n=0}^{\infty} e_n(x),$$

where

$$e_n(x) = \sum_{j=[nr]}^{[(n+1)r]-1} x^{jr'}/j!, \tag{70}$$

where $[y]$ is the greatest integer $\leqq y$. Then for $x \geqq 1$, $[nr] \leqq j \leqq [(n+1)r]-1$ we have

$$x^{jr'} \leqq x^{(n+1)rr'} = x^{n+1}$$

while

$$\begin{aligned}j! &\geqq ([nr])!\\ &\geqq \Gamma(nr)\\ &\geqq \Gamma((n+1)r+1)/((n+1)r)^{r+2}.\end{aligned}$$

Thus, for $x \geqq 1$, we have

$$\begin{aligned}e_n(x) &\leqq (r+1)x^{n+1}((n+1)r)^{r+2}/\Gamma((n+1)r+1)\\ &\leqq \beta\alpha(2x)^{n+1}/\Gamma((n+1)r+1).\end{aligned}$$

Thus,

$$\begin{aligned}\exp(x^{r'}) &= \sum e_n(x)\\ &\leqq \beta \sum (2x)^n/\Gamma(nr+1)\\ &= \beta Q(2x).\end{aligned}$$

To prove the first inequality in (70), write

$$\exp(x^{r'}) = \sum e'_n(x),$$

where

$$e'_n(x) = \sum_{j=[nr]+1}^{[(n+1)r]} x^{jr'}/j!$$

and

$$e'_0(x) = \sum_{j=0}^{[r]} x^{jr'}/j!.$$

For $[nr]+1 \leqq j \leqq [(n+1)r]$ $(n>0)$ we have

$$x^{jr'} \geqq x^{nrr'} = x^n$$

while

$$\begin{aligned}j! &\leqq \Gamma((n+1)r+1)\\ &\leqq ((n+1)r)^{r+2}\Gamma(nr+1).\end{aligned}$$

Thus,

$$\begin{aligned}\exp(x^{r'}) &= \sum e'_n(x)\\ &\geqq (r+2)\sum_{n=0}^{\infty} x^n((n+1)r)^{-r-2}/\Gamma(nr+1)\\ &\geqq \alpha \sum (x/2)^n/\Gamma(nr+1)\\ &\geqq \alpha Q(x),\end{aligned}$$

which is the desired result.

Proof of Theorem continued. Since the cases $r=1,\infty$ are known (see Ehrenpreis [8]), we may assume $r \neq 1,\infty$. Let N be any neighborhood of zero in ${}_r\mathcal{E}$ on which S is bounded. By definition, there exist sequences $\{a_n{}^k\}$ $k=1,2,\cdots,l$ of positive numbers with $a_n{}^k = O(\epsilon^n/\Gamma(rn))$ for some $\epsilon>0$ and $p>0$ so that N contains the set of $f \in {}_r\mathcal{E}$ which satisfy $\max_{|x|\leqq p} \sum a_n{}^k |f^{(n)}(x)| \leqq 1$. Our above calculations show that for some $A>0$ we have

$$\exp(-A(|Rz|^{r'} + |Iz|))\exp(iz\cdot) \in N$$

for all complex numbers z. It follows immediately that $\mathfrak{F}(S)$ satisfies (66).

To prove that, conversely, any F satisfying (66) lies in ${}_r\mathcal{E}'$, we need the following lemma which can be proved in a manner similar to the proof of the corresponding result for the space ${}_\infty\mathcal{E}$ (see Ehrenpreis [1]):

LEMMA 14. *For $r \neq 1$, ${}_r\mathcal{E}'$ consists of all $S \in {}_r\mathcal{D}'$ for which the convolution $S*f \in {}_r\mathcal{D}$ for any $f \in {}_r\mathcal{D}$. Consider ${}_r\mathcal{E}$ as a space of continuous linear maps (under convolution) of ${}_r\mathcal{D}$ into ${}_r\mathcal{D}$. Then the topology of ${}_r\mathcal{E}'$ is the compact-open topology for this set of maps.*

Now, it follows from Theorem 11′ that any function F satisfying (66) has the property that $FG \in {}_r\hat{\mathcal{D}}$ whenever $G \in {}_r\hat{\mathcal{D}}$. The fact that $f \in {}_r\hat{\mathcal{E}}'$ results from this by a simple argument (see the analogous argument in Ehrenpreis [1]).

Next, let N_H be a set as described in the statement of Theorem 16, where H satisfies (68). We want to show that N_H is a neighborhood of zero in ${}_r\hat{\mathcal{E}}'$. By a result of Grothendieck [1], ${}_r\hat{\mathcal{E}}'$ is bornologic so it is sufficient to prove that N_H swallows every bounded set, i.e., if B is bounded in ${}_r\hat{\mathcal{E}}'$ then for some $a > 0$ we have $aB \subset N_H$. This is proven by essentially the same argument as that used to prove that each $F \in {}_r\hat{\mathcal{E}}'$ satisfies (66).

The fact that the sets N_H form a fundamental system of neighborhoods of zero in ${}_r\hat{E}'$ is proven in essentially the same way as the corresponding result for the case $r = \infty$ (see Ehrenpreis [8]).

III. General remarks.

1. Let us consider, instead of the spaces ${}_r\mathcal{E}$, the spaces of functions $f \in \mathcal{E}$ for which the series $\sum a_n f^{(n)}$ converges in $\mathcal{E}$ whenever

$$a_n = O(\epsilon^n/\Gamma(rn)) \text{ for all } \epsilon > 0. \tag{71}$$

We give this space the usual topology and we obtain a topological vector space ${}^r\mathcal{E}$. In particular, ${}^1\mathcal{E}$ is the space of real analytic functions R. It seems to be a difficult task to describe the topology of the Fourier transform ${}^r\hat{\mathcal{E}}'$ of the dual of ${}^r\mathcal{E}$. One of the main difficulties is that no analog of Lemma 14 is possible for ${}^1\mathcal{E}$. On the other hand, a result Polyà (see Vl. Bernstein [1]) shows that ${}^1\mathcal{E}'$ consists of all entire functions of exponential type which are $O(\exp(\epsilon\,|\,x\,|))$ on the real axis for any $\epsilon > 0$.

2. A very difficult problem connected with the theory of infinite derivatives is the following: What kinds of conditions are necessary on the sets C_j, L_j in order that the space $\mathcal{G}$ should not be reduced to $\{0\}$. This problem is closely connected to the problems of the minimum modulus and quasi-analytic classes. Cf. Example 7 and Beurling [1].)

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ON THE HILBERT-SIEGEL MODULAR SPACE.*

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Introduction. Let $\mathfrak{S}_n$ be the space of complex, symmetric, $n \times n$ matrices $Z = X + iY$ with positive definite imaginary part Y, $Y >> 0$, and let $\mathfrak{S}_n{}^p = \mathfrak{S}_n \times \cdots \times \mathfrak{S}_n$ be the product of p replicas of $\mathfrak{S}_n$. Let $Sp(n, R)$ be the group of $2n \times 2n$ real symplectic matrices. Then $Sp(n, R)$ is naturally homomorphic to a transitive group of complex analytic transformations of $\mathfrak{S}_n$, and therefore $(Sp(n, R))^p$ is (homomorphic to) a transitive group of complex analytic transformations of $(\mathfrak{S}_n)^p$. Let $\mathfrak{k}$ be a totally real algebraic number field of (absolute) degree p and let $\mathfrak{o}$ be the ring of integers in $\mathfrak{k}$. The group $Sp(n, \mathfrak{k})$ of points in $Sp(n, R)$ having coordinates in $\mathfrak{k}$ can be imbedded in $(Sp(n, R))^p$ by

$$i: M \to (M^{\sigma_1}, \cdots, M^{\sigma_p}),$$

where $\sigma_1 =$ identity, $\sigma_2, \cdots, \sigma_p$ are the distinct isomorphisms of $\mathfrak{k}$ into R. The group $\Gamma_{n,p}{}^* = i(Sp(n, \mathfrak{k}))$ is everywhere dense in the Lie group $(Sp(n, R))^p$, and therefore the orbit of any point of $\mathfrak{S}_n{}^p$ under $\Gamma_{n,p}{}^*$ is everywhere dense in $\mathfrak{S}_n{}^p$. Let $Sp(n, \mathfrak{o})$ be the subgroup of $Sp(n, \mathfrak{k})$ consisting of those M with coordinates in $\mathfrak{o}$ such that $\det M = 1$ and put $\Gamma_{n,p} = i(Sp(n, \mathfrak{o}))$. Then $\Gamma_{n,p}$ acts in a properly discontinuous manner on $\mathfrak{S}_n{}^p$ and is known as the Hilbert-Siegel modular group [10]. The quotient space $(\mathfrak{S}_n)^p/\Gamma_{n,p}$, denoted by $\mathfrak{V}_{n,p}$, has finite volume with respect to the invariant measure on $(\mathfrak{S}_n)^p$. The same thing will be true of any subgroup $\Gamma_{n,p}{}'$ of $\Gamma_{n,p}{}^*$ commensurable with $\Gamma_{n,p}$. We note in passing that if $\gamma \in \Gamma_{n,p}{}^*$, then $\gamma^{-1}\Gamma_{n,p}\gamma$ is commensurable with $\Gamma_{n,p}$. When no confusion can arise, we shall denote $\Gamma_{n,p}{}^*$, $\Gamma_{n,p}$, and $\Gamma_{n,p}{}'$ by $\Gamma_n{}^*$, Γ_n, and $\Gamma_n{}'$ respectively, or still more simply by Γ^*, Γ, and Γ'. The main purpose of our paper will be to show that each of the quotient spaces $(\mathfrak{S}_n{}^p)/\Gamma'$ can be compactified to a normal general analytic space, isomorphic to a projective variety. With a few important exceptions, our proof is quite similar to the treatment of the Siegel modular group ($p = 1$) [12], and therefore we shall in many cases simply refer the reader to [12] for the more or less formal

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details of proofs, reserving our efforts for concentration on those aspects which apparently cannot be lifted trivially from the case of the Siegel modular group, $p = 1$.

We note here that the set of abelian varieties whose multiplication ring is an order in $\mathfrak{k}$ may be identified (non-canonically) with $(\mathfrak{S}_n)^p$ (see [10, pp. 36-42]). It is this point of view that has motivated the approach in [10]. However, we shall not pursue this interesting fact any further here. We should also like to call attention to the (well-known) fact that the group of units of $\mathfrak{o}$ plays an important role in constructing the compactifications of the spaces $(\mathfrak{S}_n)^p/\Gamma'$; it might be of interest to examine this fact further, but we cannot contribute anything in this direction at present.

At this point we introduce certain notation which we have chosen to come as close as possible to that of [12]. First of all, if σ_l is one of the isomorphisms of $\mathfrak{k}$ into R, and if x is some quantity over $\mathfrak{k}$, then x^l shall denote the result of applying σ_l to x, while if $Z \in \mathfrak{S}_n{}^p$, Z^l shall denote the l-th component of Z in the product $\mathfrak{S}_n \times \cdots \times \mathfrak{S}_n$, $z_{ij}{}^l$ denoting the ij-th entry of the matrix Z^l. Then if M denotes a p-tuple of matrices, $\mathrm{tr}(M) = \sum_l (\sum_i m_{ii}{}^l)$. If S is a matrix over k, we write $S \gg 0$ if each of the matrices $S^1, \cdots, S^p$ is positive definite, and $S \geqq 0$ if the requirement of positive definiteness is replaced by positive semi-definiteness. Moreover,

$$N \det(CZ + D) = \prod_l \det(C^l Z^l + D^l),$$

and, in general, multiplicative quantities of Siegel's theory [14], such as $\det Y$, should be replaced by the absolute norms of those quantities, whereas additive quantities such as $ds^2 = \mathrm{tr}(Y^{-1}dZY^{-1}dZ)$ should be replaced by the absolute traces of these quantities. Finally, "integer" will usually mean an element of $\mathfrak{o}$.

1. Reduction theory and fundamental open sets for Γ'. We wish to construct a fundamental open set for $\Gamma_{n,p}{}'$ in $\mathfrak{S}_n{}^p$, i.e., an open set B' in $\mathfrak{S}_n{}^p$ having finite invariant volume such that $\Gamma_{n,p}{}'B' = \mathfrak{S}_n{}^p$ and such that $\gamma B' \cap B'$ is non-empty for only a finite number of $\gamma \in \Gamma_{n,p}{}'$. For the convenience of the reader we indicate briefly the results of [8] and [10] which we shall need. Since a fundamental open set for Γ' can be constructed by taking the union of a finite number of fundamental open sets for Γ, we first construct a fundamental open set B for Γ.

Let E_{2m} be a $2m$-dimensional vector space over the rational numbers and let E_{2m} be supplied with a skew-symmetric bilinear form (x, y). The group

of linear automorphisms of E_{2m} leaving this bilinear form invariant is called the group of symplectic transformations of E_{2m} (with respect to the bilinear form). A subspace Q of E_{2m} is called a null space if $(x, y) = 0$ for all $x, y \in Q$. A linear transformation A of E_{2m} is called symplectically self-adjoint if $(Ax, y) = (x, Ay)$, $x, y \in E_{2m}$. If ${}^t\mathfrak{A}$ is an algebra of symplectically self-adjoint transformations of E_{2m}, we denote by $S_{\mathfrak{A}}$ the group of integral symplectic transformations M of E_{2m} such that $M{}^tAM = {}^tAM$ for all ${}^tA \in {}^t\mathfrak{A}$. (Here ${}^tA$ is the transpose of A and ${}^t\mathfrak{A}$ is the algebra consisting of the transposes of elements of the algebra $\mathfrak{A}$.) Two maximal null spaces Q_1 and Q_2 ($\neq 0$) of E_{2m}, invariant under ${}^t\mathfrak{A}$, are called equivalent if there exists $M \in S_{\mathfrak{A}}$ such that $MQ_1 = Q_2$.

A representation $\alpha \to \mathfrak{A}(\alpha)$ of $\mathfrak{k}$ by $m \times m$ matrices with entries from the field of rational numbers is called normal if the entries of $\mathfrak{A}(\alpha)$ are rational integers whenever $\alpha \in \mathfrak{o}$. The number of equivalence classes of normal representations (under unimodular equivalence) is finite, being bounded by $h^{m/p}$, where h is the class number of $\mathfrak{k}$. Using this fact, it is not hard to prove that if $\mathfrak{A}$ is a normal representation of $\mathfrak{k}$ by $2m \times 2m$ matrices A such that ${}^tA$ is symplectically self-adjoint, then the number of equivalence classes of maximal invariant null-spaces is finite.

An $r \times s$ matrix over $\mathfrak{k}$ may be viewed as a $pr \times ps$ matrix over the rational numbers. Let C and D be $n \times n$ matrices over $\mathfrak{o}$; the pair (C, D) is called symmetric if $C{}^tD = D{}^tC$, and primitive if for any non-singular matrix B over $\mathfrak{k}$, (BC, BD) integral implies $|N \det B| \geqq 1$. By a choice of integral basis in $\mathfrak{k}$, C and D may be viewed as $m \times m$ matrices with rational integral entries. Let L denote the m-dimensional vector sub-space of E_{2m} which the rows of (CD) span over the field of rational numbers. The condition $C{}^tD = D{}^tC$ says simply that this vector space L is a null space with respect to a certain skew-symmetric, bilinear form. The field $\mathfrak{k}$, which operates linearly on the vector space of dimension $2n$ over $\mathfrak{k}$, then possesses a natural representation as an algebra ${}^t\mathfrak{A}$ of symplectically self-adjoint transformations of E_{2m} (with respect to the above bilinear form), and $S_{\mathfrak{A}}$ is just the subgroup of the group of integral, unimodular, symplectic transformations of E_{2m} which arise in the obvious way from elements of $Sp(n, \mathfrak{o})$. The primitive pairs (C, D) and (C_1, D_1) are said to be equivalent if there exists $U \in Sp(n, \mathfrak{o})$ and a unimodular $n \times n$ matrix B (i.e., $N \det B$ is a unit), over $\mathfrak{k}$ such that $B(CD)U = (C_1D_1)$. Then it is easily seen that the primitive pairs (C, D) giving rise to the same maximal invariant null-space in E_{2m} are equivalent. Hence, there are only a finite number of equivalence classes of primitive pairs (C, D).

We now proceed ([10]) to construct a fundamental open set for $\Gamma_{n,p}$. If (C,D) is a primitive symmetric pair, define

$$\chi(C,D;Z) = |N\det(CZ+D)|.$$

If $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathfrak{o})$,

$$i(M)(Z^1,\cdots,Z^p) = ((A^1Z^1+B^1)(C^1Z^1+D^1)^{-1},\cdots,(A^pZ^p+B^p)(C^pZ^p+D^p)^{-1}).$$

Let $\chi(M;Z)=\chi(C,D;Z)$. In the future, we shall use MZ to denote $i(M)Z$. If $\mu>0$, define

$$\Omega(\mu)=\{Z \mid |\chi(C,D;Z)|>\mu \text{ for all symmetric, coprime pairs } C,D\}.$$

Using the fact that the number of equivalence classes of primitive symmetric pairs is finite, it is not hard to show that there exists $\mu_0>0$ and a finite number of transformations $M_1,\cdots,M_q \in Sp(n,\mathfrak{k})$ such that if $Z\in\mathfrak{S}_n^p$, then there exists $M\in Sp(n,\mathfrak{o})$ for which $M_\nu MZ\in\Omega(\mu_0)$ for some ν, $1\leqq\nu\leqq q$.

If $Y=(Y^1,\cdots,Y^p)$, $Y^i={}^tY^i>>0$, $i=1,\cdots,p$, then each $Y^i=D^i[T^i]$, D^i being diagonal with entries $d_{jk}{}^i=\delta_{jk}d_j{}^i$ and T^i being unipotent triangular with entries $t_{jk}{}^i$ ($t_{jk}{}^i=\delta_{jk}$ if $k\leqq j$). If $t>0$, define

$$Q(t)=\{Y \mid |t_{jk}{}^i|<t \text{ if } j>k,\ t^{-1}d_j{}^{i_1}<d_j{}^{i_2}<td_j{}^{i_1},$$
$$d_j{}^i<td_{j+1}{}^i, \text{ all } i,i_1,i_2,j\}.$$

According to Humbert [8], there exists $t_0>0$ and a finite number of non-singular matrices $A_1,\cdots,A_r\in GL(n,\mathfrak{k})$ such that every $Y=(Y^1,\cdots,Y^p)$ can be written as $Y'[A_\rho U]$ for some $Y'\in Q(t_0)$, $1\leqq\rho\leqq r$, and U belonging to the group of unimodular matrices over $\mathfrak{o}$. Choose a basis $\omega_1,\cdots,\omega_p$ of the integers in $\mathfrak{k}$ and let $Q'(t)$ denote the set of $Z\in\mathfrak{S}_n^p$ such that $Y=\operatorname{Im}Z\in Q(t)$ and such that $\operatorname{Re}Z=X=(X^1,\cdots,X^p)$, where X^i, $1\leqq i\leqq p$, can be written as $X^i=\sum_j \omega_j{}^i\Xi_j$, Ξ_j being a real symmetric matrix with entries $\xi_{j,k,l}$ satisfying $|\xi_{j,k,l}|<t$. For any μ, $t>0$ let $\Omega=\Omega_n=\Omega(\mu,t)=\Omega(\mu)\cap Q'(t)$. Then there exists a constant $\lambda>0$, $\lambda=\lambda(\mu,t)$, such that if $Z\in\Omega(\mu,t)$, we have $\operatorname{Im}Z=Y>>\lambda E$.

Then from the results of [8] and [10], and the above, it follows that there exists a set $B(\Omega)$ consisting of a finite number of translates of Ω under elements of Γ^* such that $\Gamma_{n,p}B(\Omega)=\mathfrak{S}_n^p$ and such that $B(\Omega)$ meets only a finite number of its translates under $\Gamma_{n,p}$. Since it follows from the existence of the constant λ that $B(\Omega)$ has finite invariant volume (the invariant volume element is $N(\det Y^{-n-1}dXdY)$), we see that $B(\Omega)$ is a fundamental open

set for Γ. If Γ' is commensurable with Γ, the union $B'(\Omega)$ of a finite number of translates of $B(\Omega)$ under elements of Γ is a fundamental open set for Γ'.

We choose $\mu < \mu_0$, $t > t_0$ such that for each r, $0 \leqq r \leqq n$, the open set $\Omega_r(\mu, t) = \Omega_r$ in $\mathfrak{S}_r^p$ satisfies the same conditions with respect to Γ_r as $\Omega_n(\mu, t)$ does with respect to Γ_n, so that the union $B(\Omega_r)$ of a certain finite number of translates of Ω_r under elements of $\Gamma_r{}^*$ will be a fundamental open set for Γ_r. Let $\mu_1 < \mu$, $t_1 > t$. We now define a topology on $\Omega_n(\mu_1, t_1) \cup \cdots \cup \Omega_0(\mu_1, t_1)$. If $Z_0 \in \Omega_r$, let $\mathfrak{U}_r$ be a neighborhood of Z_0 in Ω_r and let $K > 0$. Define $\mathfrak{U}_s = V^s(\mathfrak{U}_r, K)$, $r \leqq s \leqq n$, to be the set of $\begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix} \in \Omega_s(\mu_1, t_1)$ such that $Z_1 \in \mathfrak{U}_r$ and $d_{r+1}{}^i > K$, $i = 1, \cdots, p$. Then by definition, $\mathfrak{U} = \bigcup_{s=r}^{n} \mathfrak{U}_s$ is one of a basis of neighborhoods of Z_0 in $\bigcup_{s=0}^{n} \Omega_s(\mu_1, t_1)$ (it may be verified as in [12, exposé 12] that this gives rise to a legitimate topology). Then we topologize $\bigcup_{s=0}^{n} \operatorname{clos}(\Omega_s(\mu, t)) \subset \bigcup_{s=0}^{n} \Omega_s(\mu_1, t_1)$ with the subspace topology.

Let $\rho: \mathfrak{S}_n^p \to \mathfrak{S}_n^p$ be defined by

$$\rho(Z^1, \cdots, Z^p) = (-(Z^1)^{-1}, \cdots, -(Z^p)^{-1}).$$

$\rho \in \Gamma_{n,p}$. Therefore $\rho(B(\Omega))$ is also a fundamental open set for $\Gamma_{n,p}$. $\rho(\Omega_n)$ is a bounded set in the $\frac{1}{2}pn(n+1)$-dimensional space of p-tuples of complex, symmetric, $n \times n$ matrices, and its closure there, $\operatorname{clos}\rho(\Omega_n)$, is naturally homeomorphic to $\operatorname{clos}(\Omega_n) \cup \cdots \cup \operatorname{clos}(\Omega_0)$ (this follows from [12, p. 12-04, Lemma 1, 1°]).

We denote by $\mathfrak{G}_r{}^n$ the subgroup consisting of those $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathfrak{k})$ such that $A = \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}$, $B = \begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_2 \end{pmatrix}$, $C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} D_1 & D_{12} \\ 0 & D_2 \end{pmatrix}$ where A_1, B_1, C_1, D_1 are $r \times r$ and A_2, B_2, and D_2 are $(n-r) \times (n-r)$. $\mathfrak{G}_r{}^n$ may be characterized as the smallest subgroup $\mathfrak{G}$ of $Sp(n, \mathfrak{k})$ with the property that if $\{Z_\nu\}$, $\{Z_\nu'\}$ are sequences in $\operatorname{clos}\Omega_n \subset \mathfrak{S}_n^p$ with limits $Z_0 \in \operatorname{clos}\Omega_r \subset \mathfrak{S}_r^p$, $Z_0' \in \operatorname{clos}\Omega_{r'}{}^p \subset \mathfrak{S}_{r'}{}^p$ such that $Z_\nu = MZ_\nu'$, $M \in \Gamma_{n,p}$ fixed, then $r = r'$ and $M \in \mathfrak{G}$ (see [12, p. 12-04, Lemma 1]). It is easily seen that $\pi: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ is a homomorphism of $\mathfrak{G}_r{}^n$ onto $\Gamma_{r,p}{}^*$. We denote the kernel of π by $\mathfrak{N}_r{}^n$. $\mathfrak{N}_r{}^n$ consists of all $M \in Sp(n, \mathfrak{k})$ of the form $M = \begin{pmatrix} {}^tU & TU^{-1} \\ 0 & U^{-1} \end{pmatrix}$, where $U \in GL(n, \mathfrak{k})$ is of the form $\begin{pmatrix} E & U_{12} \\ 0 & U_2 \end{pmatrix}$ and where T is a symmetric $n \times n$ matrix over $\mathfrak{k}$ of the form $T = \begin{pmatrix} 0 & T_{12} \\ {}^tT_{12} & T_2 \end{pmatrix}$. If M

belongs to a subgroup of $Sp(n, \mathfrak{k})$ commensurable with $Sp(n, \mathfrak{o})$, it is easy to see that some power of U must be in the group of unimodular matrices over $\mathfrak{o}$, and therefore $N \det U = \pm 1$.

If $\mathfrak{G}$ is a group of transformations of the space D_1 and τ a homeomorphism of D_1 onto the space D_2, we let $\mathfrak{G}^\tau = \tau \mathfrak{G} \tau^{-1}$.

By the mapping

$$\tau : (Z^j) \to ((E + iZ^j)(E - iZ^j)^{-1})$$

(the so-called "Cayley transformation") $\mathfrak{S}_n{}^p$ is transformed onto $\mathfrak{B}_n{}^p$, where $\mathfrak{B}_n$ is the "generalized unit disc" consisting of all symmetric complex matrices W such that $W\bar{W} << E$, and $\Gamma_{n,p}$ is transformed into the properly discontinuous group $\Gamma_{n,p}{}^\tau$ acting on $\mathfrak{B}_n{}^p$. $\tau(\rho(B(\Omega)))$ is evidently a fundamental open set for $\Gamma_{n,p}{}^\tau$. $Sp(n, \mathfrak{k})^\tau$ is an almost transitive (i.e., the orbit of each point is dense) group of transformations of $\mathfrak{B}_n{}^p$, and is also a group of continuous transformations of $\operatorname{clos}(\mathfrak{B}_n{}^p) = \mathfrak{A}_n{}^p$. As in [12, p. 12-15] we let

$$\mathfrak{A}_{n,r}{}^p = \{(W_j) \mid \operatorname{rank}(E - W_j \bar{W}_j) \leqq r\}$$
$$\mathfrak{B}_{n,r}{}^p = \{(W_j) \mid \operatorname{rank}(E - W_j \bar{W}_j) = r\}.$$

Each $\mathfrak{B}_{n,r}{}^p$ is stable under $Sp(n, \mathfrak{k})^\tau$, and the latter acts almost transitively on $\mathfrak{B}_{n,r}{}^p$. As in [12, p. 13-01] we may easily prove that

$$Sp(n, \mathfrak{k})^\tau \operatorname{clos}(\tau(\rho(B(\Omega)))) = \Gamma_{n,p}{}^\tau \operatorname{clos}(\tau(\rho(B(\Omega))))$$

(the operations of closure being with respect to the entire space of $n \times n$ complex, symmetric matrices). We denote $\Gamma_{n,p}{}^\tau \operatorname{clos}(\tau(\rho(B(\Omega))))$ by $\mathfrak{A}_n{}^*$ (there is no need to exhibit p explicitly here since the field $\mathfrak{k}$ is fixed throughout). We denote $\tau(\rho(\Omega))$ by Σ and $\tau(\rho(B(\Omega)))$ by $B(\Sigma)$. Then $\operatorname{clos}(B(\Sigma))$ is the union of a finite number of translates of $\operatorname{clos}(\Sigma)$ by elements of $Sp(n, \mathfrak{k})^\tau$, $\operatorname{clos}(B(\Sigma)) = \bigcup_{i=1}^{s} \sigma_i \operatorname{clos}(\Sigma)$, $\sigma_i \in Sp(n, \mathfrak{k})^\tau$. We topologize $\operatorname{clos}(B(\Sigma))$ with the finest topology such that each of the mappings

$$\operatorname{clos}(\Sigma) \to \sigma_i \operatorname{clos}(\Sigma) \subset \operatorname{clos}(B(\Sigma))$$

is continuous, and topologize $\mathfrak{A}_n{}^*$ with the finest topology such that each of the mappings

$$\operatorname{clos}(B(\Sigma)) \to \gamma \operatorname{clos}(B(\Sigma)) \subset \mathfrak{A}_n{}^*,$$

$\gamma \in Sp(n, \mathfrak{k})$, is continuous. Then as in [12, exposés 12 and 13] it is not

hard to see that $Sp(n,\mathfrak{k})^\tau$ acts as a group of continuous transformations of $\mathfrak{A}_n^*$. We let (as in [12, p. 12-15]) [1]

$$\mathfrak{A}_{n,r}^* = \mathfrak{A}_n^* \cap \mathfrak{A}_{n,r}, \qquad \mathfrak{B}_{n,r}^* = \mathfrak{B}_{n,r} \cap \mathfrak{A}_n^*$$

$0 \leqq r \leqq n$. As previously remarked, the homeomorphism ρ of Ω_n onto $\rho(\Omega_n)$ may be extended to a homeomorphism, which we again denote by ρ, of $\bigcup_{s=0}^{n} \operatorname{clos}(\Omega_s)$ onto $\operatorname{clos}\rho(\Omega_n)$. Let C_r denote the connected component of $\mathfrak{B}_{n,r}^*$ containing $\tau(\rho(\Omega_r))$; then C_r is complex analytically isomorphic to $\mathfrak{S}_r^p$, the subgroup $(\mathfrak{G}_n^r)^\tau$ of $Sp(n,\mathfrak{k})^\tau$ acts almost transitively on C_r, and if $\sigma \in Sp(n,\mathfrak{k})^\tau - (\mathfrak{S}_n^r)^{\tau\rho}$, $C_r \cap \sigma C_r$ is empty. Moreover, if $\gamma \in \mathfrak{G}_n^r$, the action of $\tau\rho\gamma(\tau\rho)^{-1}$ on C_r is the same as that of $\pi_r(\gamma)$ on $\mathfrak{S}_r^p$ under our identification of C_r with $\mathfrak{S}_r^p$. These facts are verified by applying Lemma 1, p. 12-04 of [12] to each factor of $\mathfrak{S}_n \times \cdots \times \mathfrak{S}_n$ and by applying considerations similar to those of pp. 12-14 to 12-15 of [12]. If $\sigma \in Sp(n,\mathfrak{k})^\tau$, we have natural isomorphisms of quotient spaces:

$$\sigma C_r/((\mathfrak{G}_n^r)^{\sigma\tau\rho} \cap \Gamma_{n,p}{}'^{\tau}) \cong C_r/((\mathfrak{G}_n^r)^{\tau\rho} \cap \Gamma_{n,p}{}'^{\sigma^{-1}\tau}) \cong \mathfrak{S}_r^p/\Gamma_{r,\lambda(\sigma)},$$

where $\Gamma_{r,\lambda(\sigma)}$ is a certain subgroup of $Sp(r,\mathfrak{k})$ commensurable with $\Gamma_{r,p}$. Although $\mathfrak{A}_r^*/\Gamma_{r,\lambda(\sigma)}{}^\tau$ may be mapped in a continuous, proper manner onto $\operatorname{clos}(C_r)/((\mathfrak{G}_n^r)^{\tau\rho} \cap \Gamma_{n,p}{}'^{\sigma^{-1}\tau})$, this mapping ψ is not a homeomorphism since [12, pp. 13-05 to 13-06] it is not even one-to-one. However, if

$$x \in \operatorname{clos}(C_r)/((\mathfrak{G}_n^r)^{\tau\rho} \cap \Gamma_{n,p}{}'^{\sigma^{-1}\tau}),$$

$\psi^{-1}(x)$ is a finite set and ψ restricted to $\mathfrak{B}_r^p/\Gamma_{r,\lambda(\sigma)}{}^\tau$ is actually one-to-one.

The finite number of points in $\mathfrak{B}_{n,0}{}^*/\Gamma_{n,p}{}'^{\tau}$ correspond to the finite number of points in which a true fundamental domain (not just a fundamental open set) meets the distinguished boundary of $\mathfrak{B}_n^p$, and their number can be calculated as in [10].

Just as in [12, exposés 12 and 13] it may be proved that $\mathfrak{A}_n^*/\Gamma_{n,p}{}'^{\tau}$ is a compact Hausdorff space. It is also easy to see that $\mathfrak{A}_n^*/\Gamma_{n,p}{}'^{\tau}$ is the union of a finite number of pairwise disjoint subspaces isomorphic to complex analytic spaces $\mathfrak{S}_r^p/\Gamma_{r,\lambda}$, $0 \leqq r \leqq n$. We shall now elaborate somewhat on the geometrical situation involved.

As we have just seen, $(\tau\rho)^{-1}$ maps $\mathfrak{B}_n^p$ onto $\mathfrak{S}_n^p$, and may be uniquely extended to a homeomorphism ϕ of $\mathfrak{A}_n^*$ with the union $\mathfrak{S}_n^*$ of a certain

[1] It should be noted as a direct consequence of the definition of $Q(t)$ (namely that certain diagonal entries of Y must be of the same order of magnitude) that $\mathfrak{A}_n^* = \bigcup_r \mathfrak{B}_{n,r}^*$.

countable collection of replicas $\mathfrak{S}_{r,\alpha^p}$ of the spaces $\mathfrak{S}_r{}^p$ $(0 \leqq r \leqq n)$ supplied with the finest topology such that each of the injections

$$\gamma(\bigcup_{s=0}^{n} \operatorname{clos} \Omega_s) \subset \mathfrak{S}_n{}^*, \qquad \gamma \in Sp(n, \mathfrak{k}),$$

is continuous, the action of $\gamma \in Sp(n, \mathfrak{k})$ on $\mathfrak{S}_n{}^*$ being defined in an evident manner. C_r is to be identified with one of the $\mathfrak{S}_{r,\alpha^p}$, which we denote simply by $\mathfrak{S}_r{}^p$, and Ω_r is an open subset of $\mathfrak{S}_r{}^p$.

Let $\mathfrak{U}$ be an open subset of $\mathfrak{S}_n{}^*$ such that $\Gamma'\mathfrak{U} = \mathfrak{U}$, and let $\mathfrak{U}_r = \mathfrak{S}_r{}^p \cap \mathfrak{U}$. It is clear that the closure of $\mathfrak{U}_n$ in the topology of $\mathfrak{S}_n{}^*$ contains $\mathfrak{U}_r$ for $r < n$. Letting

$$Sp(n, \mathfrak{k}) = \bigcup_{\lambda} \Gamma' M_{r\lambda} \mathfrak{G}_r{}^n$$

be a decomposition of $Sp(n, \mathfrak{k})$ into double cosets mod Γ' and $\mathfrak{G}_r{}^n$ (which, as in [12, exp. 13], may be seen to be finite in number), put

$$\mathfrak{U}_{r\lambda} = M_{r\lambda}{}^{-1}\mathfrak{U} \cap \mathfrak{S}_r{}^p$$

and

$$\Gamma_{r\lambda} = \pi_r(M_{r\lambda}{}^{-1}\Gamma' M_{r\lambda} \cap \mathfrak{G}_r{}^n).$$

Then $\mathfrak{U} = \bigcup_{r,\lambda} \Gamma' M_{r\lambda}\mathfrak{U}_{r\lambda}$, and $(\Gamma' M_{r\lambda}\mathfrak{U}_{r\lambda}) \cap (\Gamma' M_{s\mu}\mathfrak{U}_{s\mu})$ is non-empty only if $r = s$, $\lambda = \mu$. If furthermore $s < r < n$ and

$$Sp(r, k) = \bigcup_{\nu} \Gamma_{r\lambda} M_{s\nu}{}^{r\lambda} \mathfrak{G}_s{}^r,$$

and if $M_{r\lambda} M_{s\nu}{}^{r\lambda} = M' M_{s\mu} L$, $M' \in \Gamma'$, $L \in \mathfrak{G}_s{}^n$, we write $(\lambda, \nu) \to \mu$. It is of course possible that we may have $(\lambda', \nu') \to \mu$, $\lambda' \neq \lambda$, $\nu' \neq \nu$.

The above has the following geometrical significance: A "fundamental open set" [2] $B^*(\Omega)$ for Γ' in $\mathfrak{S}_n{}^*$ consists of the interior of the closure in $\mathfrak{S}_n{}^*$ of the fundamental open set $B'(\Omega)$ for Γ' in $\mathfrak{S}_n{}^p$. For each r, $B^*(\Omega)$ meets a certain finite number of the spaces $\mathfrak{S}_{r,\alpha^p}$. Let a maximal subset of these, inequivalent under Γ', be denoted by $\mathfrak{S}_{r\lambda}{}^p$, λ belonging to a finite indexing set Λ, and we may choose the $M_{r\lambda}$ such that $M_{r\lambda}\mathfrak{S}_r{}^p = \mathfrak{S}_{r\lambda}{}^p$. Let $\delta(r) = \frac{1}{2}pr(r+1)$. Then $M_{r\lambda}$ carries the $\delta(r)$-dimensional "vertex" $M_{r\lambda}{}^{-1} B^*(\Omega) \cap \mathfrak{S}_r{}^p$ onto the $\delta(r)$-dimensional vertex $B^*(\Omega) \cap \mathfrak{S}_{r\lambda}{}^p$ of $B^*(\Omega)$. Moreover, $B_{r\lambda}{}^*(\Omega) = M_{r\lambda}{}^{-1}(B^*(\Omega) \cap \mathfrak{S}_{r\lambda}{}^p)$ is a fundamental open set in $\mathfrak{S}_r{}^p$ for the group $\Gamma_{r\lambda}$, and $M_{s\nu}{}^{r\lambda}$ may be chosen to carry the $\delta(s)$-dimensional

[2] Here the use of the term "fundamental open set" is a convenient abuse of language (it is inaccurate because Γ' does not act discontinuously in $\mathfrak{S}_n{}^*$). The term is used here to denote a set containing only a finite number of points from each orbit of Γ' and whose intersection with each of the lower dimensional spaces $\mathfrak{S}_{r,\alpha^p}$ is a fundamental open set in this space with respect to a certain group acting there.

vertex $B_{r\lambda}^*(\Omega) \cap \mathfrak{S}_s^p$ onto the $\delta(s)$-dimensional vertex $M_{sv}^{r\lambda}(B_{r\lambda}^*(\Omega) \cap \mathfrak{S}_s^p)$ of $B_{r\lambda}^*(\Omega)$, and so $M_{r\lambda}M_{sv}^{r\lambda}$ carries the vertex $B_{r\lambda}^*(\Omega) \cap \mathfrak{S}_s^p$ (which is transformed onto some other open domain in $\mathfrak{S}_s^p$ by all $\gamma \in \mathfrak{G}_s^n$) onto some other $\delta(s)$-dimensional space which is a translate under some $M' \in \Gamma'$ of a $\delta(s)$-dimensional vertex contained in, say, $\mathfrak{S}_{s\mu}^p$.

Let $Z_0 \in \Omega_r$, let $\mathfrak{U}_r$ be a connected neighborhood of Z_0, stable under $\pi_r(\Gamma')_{Z_0}$, and let $\mathfrak{U}_s = V^{(s)}(\mathfrak{U}_r, K)$ be defined as before, $r < s \leqq n$, $K > 0$. It is easy to see that there exist a finite number of $M \in (\Gamma^*)_{Z_0}$, say $M_1, \cdots, M_m$, such that the closure of $(\Gamma')_{Z_0}(\bigcup_{i=1}^{m} M_i\mathfrak{U}_n)$ is a (saturated) neighborhood of Z_0 (in $\mathfrak{A}_n^*$, if we identify $\mathfrak{S}_n^p$ with $\mathfrak{B}_n^p$). However, $\bigcup_{1}^{m} M_i\mathfrak{U}_n$ may not be connected. Let $N_1, \cdots, N_k \in Sp(n, \mathfrak{k})_{Z_0}$ be of the form

$$N_j = \begin{pmatrix} {}^tU_j & 0 \\ 0 & U_j^{-1} \end{pmatrix}, \quad \text{where } U_j = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & e_j \end{bmatrix},$$

and $e_1, \cdots, e_k$ have all possible signatures in $\mathfrak{k}$. Let

$$\left\{ \begin{bmatrix} Z_0^{(l)} & & & 0 \\ & y_{r+1}^{(l)} & & \\ & & \ddots & \\ 0 & & & y_n^{(l)} \end{bmatrix} \right\}_{l=1,\cdots,p} = Z_0^* \in \mathfrak{U}_n.$$

For appropriate *fixed* choice of μ, t, we can always find $y_{r+1}^{(l)}, \cdots, y_n^{(l)}$ (depending on $\mathfrak{U}_n$) such that the "ray" from Z_0^* to Z_0 will be containend in $\mathfrak{U}_n \cap N_j\mathfrak{U}_n$, $j = 1, \cdots, k$. Let G_{Z_0} be the subgroup of $Sp(r, R)^p$ leaving Z_0 fixed. Then the kernel $\mathfrak{N}$ of the canonical homomorphism of $(Sp(n, R)^p)_{Z_0}$ onto G_{Z_0} has as many components as there are different signatures (i.e., 2^p). Therefore if $M_i \in g\mathfrak{N}$, M_i can be joined to one gN_j by a fixed path $\mathfrak{K}_i$ in $g\mathfrak{N}$. The union $\mathfrak{K} = \bigcup_i \mathfrak{K}_i$ is a compact set fixed once for all. Let $\mathfrak{L}$ be the (dense) set of points of $iSp(n, \mathfrak{k})$ contained in a fixed compact neighborhood $\mathfrak{R}$ of $\mathfrak{K}$ in the space consisting of the finite number of cosets $g\mathfrak{N}$ involved. Let $D = (\Gamma')_{Z_0}\mathfrak{L}\mathfrak{U}_n$. Since $\mathfrak{U}_r$ is connected, it follows easily from our construction that D is connected. Since each $g \in Sp(n, \mathfrak{k})$ acts continuously on $\mathfrak{A}_n^*$ (with our usual identifications), it is clear that the closure of D is a neighborhood of Z_0. Finally, direct computation shows that, since $\mathfrak{R}$ is *compact* and fixed once for all, the closure of D runs over a basis of neighborhoods of Z_0 as $\mathfrak{U}_r$ runs over a basis of connected neighborhoods of Z_0 in $\mathfrak{S}_r$

and $K \to +\infty$. Therefore, since D is connected for each of these, we have (as in [12, pp. 13-08 to 13-10]): *the canonical image of* Z_0 *in* $\mathfrak{S}_n^*/\Gamma'$ *has a basis of neighborhoods* $\{U_\alpha\}$ *such that each* $U_\alpha \cap (\mathfrak{S}_n^p/\Gamma')$ *is connected.*

2. Modular forms and Eisenstein series. Let Γ' be a subgroup of $Sp(n, \mathfrak{k})$ commensurable with the modular group $\Gamma = \Gamma_{n,p}$. By a (matrix-valued) modular form of weight k and multiplier v with respect to Γ', we mean a holomorphic (matrix-valued) function f on $\mathfrak{S}_n^p$ with values in the space $M_q(C)$ of $q \times q$ complex matrices satisfying

$$f(MZ) = N \det (CZ + D)^k f(Z) v(M)^{-1}$$

for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma'$, where v is a homomorphism of Γ' into $GL(q, C)$ such that the kernel of v is of finite index in Γ'. We denote the module of such forms by $\mathfrak{H}_{n,\Gamma'}(k, v)$, and if $q = 1$ and $v(M) = 1$ for all $M \in \Gamma'$, this module is denoted by $\mathfrak{H}_{n,\Gamma'}(k)$.

Let S be an $n \times n$ symmetric matrix over $\mathfrak{o}$, and assume $S \geqq 0$, which means that each of the conjugates of S is positive semi-definite, all conjugates, of course, being of the same rank r. As in [12] we let

$$\mathfrak{G}_S = \{M \mid M \in \mathfrak{G}_0^n, \quad M = \begin{pmatrix} {}^tU & TU^{-1} \\ 0 & U^{-1} \end{pmatrix}, \quad US\,{}^tU = S, \quad \mathrm{tr}(ST) \in \mathfrak{o}\},$$

$$\Gamma_S' = \Gamma' \cap \mathfrak{G}_S.$$

The Eisenstein series $E_{\Gamma';S,k}$ for an even integer k, Γ', and S is defined by

$$E_{\Gamma';S,k}(Z) = \sum_{M:\,\Gamma_S' \backslash \Gamma'} \epsilon(S \cdot MZ) N \det (CZ + D)^{-k},$$

where $\epsilon(X) = \exp(2\pi i t r(X))$, and where $M: \Gamma_S' \backslash \Gamma'$ means that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs over a complete system of right coset representatives of Γ_S' in Γ'. For fixed k, let $E_{\Gamma';S} = E_{\Gamma';S,k}$.

We shall shortly return to the proof of the convergence of these series for sufficiently large k, noting that is sufficient to prove the convergence of $E_{\Gamma;S}$ because of the following easy propositions:

(I) If Γ'' is a subgroup of Γ', and if $E_{\Gamma';S}$ converges, then $E_{\Gamma'';S}$ converges.

(II) If Γ'' is a normal subgroup of Γ' of finite index, the convergence of $E_{\Gamma'';S}$ implies that of $E_{\Gamma';S}$.

It is easily seen that $E_{\Gamma';S}$ is a modular form of weight k with respect

to Γ'. Moreover, if we define $(f|M)(Z) = f(MZ)N\det(CZ+D)^{-k}$ for any $M \in Sp(n,\mathfrak{k})$, it is clear that

$$E_{M^{-1}\Gamma' M} \mid M \in \mathfrak{H}_{n,\Gamma'}(k).$$

Let $\omega \in GL(q,C)$ be fixed and let

$$\Gamma'_{S,\omega} = \{M \mid M = \begin{pmatrix} {}^tU & TU^{-1} \\ 0 & U^{-1} \end{pmatrix} \in \Gamma' \cap \mathfrak{G}_0^n;$$

$$\epsilon(SZ)\omega \mid M = \epsilon(SZ)\omega\},$$

where $\epsilon(SZ)\omega \mid M = \epsilon(SZ)N\det(CZ+D)^k \omega v(M)$ for $M \in Sp(n,\mathfrak{k})$. Then more generally we can consider the series

$$E_{\Gamma';S,\omega}(Z) = \sum_{M:\Gamma'_{S,\omega}\backslash\Gamma'} \epsilon(SZ)\omega|M,$$

which, if it converges, $\in \mathfrak{H}_{n,\Gamma'}(k,v)$. In order to establish the convergence of this it is evidently sufficient to prove the convergence of the series $E_{\Gamma;S}$.

We first consider the case $S \gg 0$. We shall not exhibit many details here because most of these are the same as in [12, exposé 9].

THEOREM 1. *Let S be an $n \times n$ symmetric matrix over $\mathfrak{k}$, $S \gg 0$, and let $E_{\Gamma;S}$ be defined as above. If $k > 2n$, $E_{\Gamma;S}$ converges uniformly on every compact subset of $\mathfrak{S}_n^p$.*

Proof. Proceeding as in [12], we let $J(M,Z) = N(\det(CZ+D))^k$, $I(Z) = N(\det Y)^{k/2}$. Then letting $J'(M,Z) = I(MZ)J(M,Z)I(Z)^{-1}$, we see that $|J'(MZ)| = 1$. Therefore if we let $\phi(Z) = I(Z)\epsilon(SZ)$, we have

$$(N(\det Y)^{k/2})E_{\Gamma;S}(Z) = \sum_{M:\Gamma_S\backslash\Gamma} J'(M,z)^{-1}\phi(MZ),$$

and since for holomorphic functions convergence in the $\mathfrak{L}^1$-norm on an open set implies uniform point-wise convergence on every compact subset, it will suffice to prove the convergence of the integral

$$\int_{Z \bmod \Gamma} \sum_{M:\Gamma_S\backslash\Gamma} \| \phi(MZ) \| \, dZ,$$

where $dZ = N((\det Y)^{-n-1}\, dX dY)$ is the invariant measure on $\mathfrak{S}_n^p$. This integral is equal to

$$\int_{Z \bmod \Gamma_S} \| \phi(Z) \| \, dZ,$$

and since the subgroup Γ_∞ of Γ_S consisting of the translations $M = \begin{pmatrix} E & T \\ 0 & E \end{pmatrix}$

is of finite index in Γ_S (since $US^tU = S$ must hold for each of the *conjugates* of U and S), it is therefore sufficient to prove the convergence of

$$\int_{Z \bmod \Gamma_\infty} \| \phi(Z) \| \, dZ$$

$$= \int_{Z \bmod \Gamma_\infty} N(\det Y)^{k/2} \exp(-2\pi tr(SY)) N(\det Y^{-n-1} \, dX dY)$$

$$= \Delta^{n(n+1)/2} \int_{\substack{Y^i >> 0 \\ i=1,\cdots,p}} \prod_i (\det Y^i)^{k/2} \exp\Big(-2\pi \sum_i tr(S^i Y^i)\Big) \prod_i (\det Y^i)^{-n-1} \, dY^i$$

$$= \Delta^{n(n+1)/2} \prod_{i=1}^{p} \int_{Y^i >> 0} (\det Y^i)^{k/2-n-1} \exp(-2\pi tr(S^i Y^i)) \, dY^i,$$

where Δ is the discriminant of $\mathfrak{k}$. Since each of these last factors converges if $k > 2n$, the proof is complete.

Let Γ' be commensurable with Γ. A norm $\| \ \|_\beta$ is defined on $\mathfrak{H}_{n,\Gamma'}(k)$ by:

$$\| f \|_\beta = \Big(\int_{Z \bmod \Gamma'} | N \det Y^{k/2} \cdot f |^\beta \, dZ \Big)^{1/\beta}.$$

The space of $f \in \mathfrak{H}_{n,\Gamma'}(k)$ with $\| f \|_\beta < +\infty$ is denoted by $\mathfrak{H}_{\Gamma',\beta}(k)$. The integral estimate of the preceding proof shows that $E_{\Gamma';S} \in \mathfrak{H}_{\Gamma',1}(k)$ if $k > 2n$.

We turn to the proof that $E_{\Gamma,S}$ converges if $S \geqq 0$. This is more difficult than the case $S >> 0$ and we shall give an arithmetic type of proof along the lines of [4, 13], rather than attempt to carry over the proof in [12]. We have

$$E_{\Gamma,S} = \sum_{M : \Gamma_S \backslash \Gamma} \epsilon(S \cdot MZ) N \det(CZ + D)^{-k}.$$

By using the device of [4, p. 391] it is easy to see that we may restrict ourselves to investigating the sum for $\det C \neq 0$. For a fixed symmetric pair (C, D), $\det C \neq 0$, we wish to estimate the sum

$$\phi_{C,D}(Z) = \sum_M \epsilon(S \cdot MZ),$$

where M runs over the same system of coset representatives for the fixed pair (C, D) as in $E_{\Gamma,S}$.

All such representatives can be chosen in the form $\begin{pmatrix} {}^tU & TU^{-1} \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, A, B being fixed once for all. Then

$$| \phi_{C,D}(Z) | \leqq \sum_U \exp(-2\pi tr(S^i \cdot Y_1{}^i[U^i])),$$

where

$$Y_1 = \mathrm{Im}\left((A^iZ^i + B^i)(C^iZ^i + D^i)^{-1}\right)$$
$$= {}^t(C^i\bar{Z}^i + D^i)^{-1}Y^i(C^iZ^i + D^i)^{-1}(Y^i = \mathrm{Im}\, Z^i)$$

and the summation is over a set of coset representatives of the group of units of S in the group of unimodular matrices over $\mathfrak{o}$ (i.e., $\det U$ is a unit of $\mathfrak{o}$). If S is of rank r, this summation is over a certain group of real lattice points in $C^{nr} \times \cdots \times C^{nr} = C^{nrp}$, with appropriate choice of coordinates (these are not products of lattice points in the component factors, but are in correspondence with a certain additive group of $r \times n$ matrices over $\mathfrak{o}$ of finite index I in the group of all such matrices in such a way that the i-th component of the lattice point is the i-th conjugate of the matrix). If

$$V = (V^1, \cdots, V^p) \in C^{nrp},$$

V^i being an $r \times n$ matrix, $(CZ + D)^{-1}$ acts as a linear transformation on C^{nrp} by

$$(CZ + D)^{-1}V = \left((C^1Z^1 + D^1)^{-1}V^1, \cdots, (C^pZ^p + D^p)^{-1}V^p\right),$$

and the *real* determinant (i.e., Jacobian) of this transformation is $|N\det(CZ + D)|^{-2r}$. To estimate $|\phi_{C,D}(Z)|$ for Z belonging to a compact subset of $\mathfrak{S}_n^p$, we sum over the lattice points in an expanding sequence of concentric ellipsoidal shells. Namely, if R is the region between two concentric sphere in C^{nrp}, with respect to the metric $tr(\sum_{i=1}^{p} {}^t\bar{V}^iV^i)$, we wish to estimate the number of lattice points of our summation in the ellipsoidal shell which is the image of R under the mapping $(CZ + D)^{-1}$. We have

$$(C^iZ^i + D^i)^{-1}V^i = (Z^i + (C^i)^{-1}D^i)^{-1}(C^i)^{-1}V^i,$$

and $I(\det C^i)(C^i)^{-1}$, having coefficients in $\mathfrak{o}$, is an endomorphism of the set of lattice points, whereas the fact that Z remains in a compact set makes it clear that the ratio of the maximum to the minimum eigenvalue of the transformation $Z + (C)^{-1}D$ is no greater than some constant times $|N\det(Z + (C)^{-1}D)|^{e_1}$, where e_1 is some positive integer. Therefore, the number of lattice points in the ellipsoidal shell is no greater than some constant times $|N(\det(CZ + D))|^{e_1}$, for a suitable (possibly altered) choice of the positive integer e_1. Since

$$|\phi_{C,D}(Z)| \leqq \sum_U \exp(-2\pi tr(S^iY^i[(C^iZ^i + D^i)^{-1}U^i]),$$

it is then clear that

$$\sup_{Z \in K} |\phi_{C,D}(Z)| \leqq \bar{C}_1 |N\det(CZ + D)|^{e_1},$$

C_1 being a positive constant depending only on the compact set K. Therefore it suffices to prove the convergence of

$$\psi(Z) = \sum_{(C,D),\,\det C\neq 0} |N\det(CZ+D)|^{-(k-e_1)},$$

(C,D) running over a system of left non-associate, symmetric, primitive pairs. We have, letting $k-e_1=\rho$,

$$\psi(Z) = \sum_{C} |N\det C|^{-\rho}\Big(\sum_{D} |N\det(Z+C^{-1}D)|^{-\rho}\Big).$$

Let $\beta_C(Z) = \sum_{D} |N\det(Z+C^{-1}D)|^{-\rho}$, D running over integral matrices such that (C,D) is symmetric.

We say that two matrices H_1 and H_2 are congruent mod C if $H_1-H_2 = C\cdot T$, T being integral. It is clear that the number of residue classes mod C of integral D such that $C^{-1}D$ is symmetric is not greater than $|N\det C|^{\frac{1}{2}n(n+1)}$. It is also clear that a representative D_0 from each residue class can be chosen such that $C^{-1}D_0$ lies in a fixed compact subset of the space of real symmetric matrices, independent of C and the particular residue class. Therefore, to prove that $\psi(Z)$ converges uniformly on K, it is sufficient to show that

$$\sum_{C} |N\det C|^{-\rho+\frac{1}{2}n(n+1)} \text{ and } \sum_{H} |N\det(Z+H)|^{-\rho}$$

converge uniformly, where in the second sumation H runs over all symmetric matrices over $\mathfrak{o}$.

If C is a non-singular matrix over $\mathfrak{o}$, the number of left non-associate matrices A over $\mathfrak{o}$ satisfying ${}^tAA = {}^tCC$ is no greater than some constant times $N\det({}^tCC)^{e_2}$ for some positive integer e_2 independent of C. In fact, if ${}^tAA = {}^tCC$, $M = (\det C)(AC^{-1})$ is a matrix over $\mathfrak{o}$ such that ${}^tMM = (\det C)^2E$, and since, after multiplying C by a suitable unit of $\mathfrak{o}$, we may assume $|(\det C^i)/(\det C^j)| < \lambda$ for all i, j, our assertion follows from a direct computation (here λ is some constant >1 depending only on the field $\mathfrak{k}$). Since ${}^tCC = S$ is a positive definite matrix with coefficients in $\mathfrak{o}$, we see that

$$\sum_{C} |N\det C|^{-\rho+\frac{1}{2}n(n+1)}$$

converges provided that the following series converges:

$$\sum_{S>>0} |N\det(S)|^{-\sigma},$$

where $\sigma = \frac{1}{2}\rho - n(n+1)/4 - e_2$ and where S runs over a system of representatives of equivalence classes of positive definite symmetric matrices over $\mathfrak{o}$

($S_1 \sim S_2$ if there exists a unimodular $n \times n$ matrix U over o such that ${}^tUS_1U = S_2$). By Humbert's reduction theory [8], there exists a finite number of matrices $A_1, \cdots, A_m \in GL(n, \mathfrak{k})$ such that for each symmetric $S >> 0$ there exists μ, $1 \leqq \mu \leqq m$, such that $S[A_\mu]$ is equivalent to a point in the domain $Q(t)$ (q. v., § 1). Therefore the series $\sum_{S>>0} |N \det S|^{-\sigma}$ is not greater than $K_1 \cdot \sum_{S \in Q(t)} |N \det S|^{-\sigma}$, K_1 being a positive constant. If $S \in Q(t)$, $S^{(i)} = (s_{jk}^{(i)})$, it is clear that $\det S^{(i)} \geqq C s^{(i)} \cdots s_{nn}^{(i)}$ and $C \cdot |s_{jk}^{(i)}| \leqq s_{jj}^{(i)}$, all i, j, k, where C is a suitable positive constant and $C s_{jj}^{(i_1)} < s_{jj}^{(i_2)} < C^{-1} s_{jj}^{(i_1)}$ all i_1, i_2, j. Let $N s_{jj} = \tau_j$. For fixed positive rational integers $\tau_1, \cdots, \tau_n$ all the elements in the j-th column of S are such that they, together with all of their conjugates are no greater than $c' \tau_j^{1/p}$. Therefore each of them can take on at most $c'' \tau_j$ values, where c'' depends only on c' and k. Hence, the number of S for given $\tau_1, \cdots, \tau_n$ is at most $c'' \cdot \tau_1 \tau_2^2 \cdots, \tau_n^n$, and therefore $\sum_{S \in Q(t)} |N \det S|^{-\sigma}$ is majorized by $K' \cdot \sum_{\tau_1, \cdots, \tau_n} \tau_1^{1-\sigma} \cdots \tau_n^{n-\sigma}$ which evidently converges if $\sigma > n+1$.

We now need only show that

$$\sum_H |N \det(Z+H)|^{-\sigma}$$

converges uniformly over every compact for sufficiently large σ. If $Z = X + iY$, we may write (as in [13]) $\mathfrak{P}'Y\mathfrak{P} = E$, $\mathfrak{P}'(X+H)\mathfrak{P} = W$. If $\lambda_1, \cdots, \lambda_n$ are the characteristic roots of W,

$$|\det(Z+H)| = \det Y \cdot \prod_{k=1}^{n} (1+\lambda_k^2)^{\frac{1}{2}} \geqq \det Y \cdot (1 + \sum_{k=1}^{n} \lambda_k^2)^{\frac{1}{2}}.$$

If we put $\Lambda^2 = \sum_{k=1}^{n} \lambda_k^2$, then $\Lambda^2 = \operatorname{tr}(W^2) = \sum_{i,j} w_{ij}^2$, so that if q is the integer such that $q - 1 \leqq 2\Lambda < q$, we have $|w_{ij}| < q$, all i, j.

$$H = (P')^{-1} W P^{-1} - X$$

and since P and X range over fixed compact sets, this implies the existence of a constant $c > 0$ such that $|h_{ij}| < cq$. Hence, for fixed q, the total number of matrices H such that $q - 1 \leqq (\operatorname{tr}((W^i)^2)) < q$ for each of the conjugates H^i of H is no greater than $c'(q^p)^{\frac{1}{2}n(n+1)}$, and for each such matrix $|N \det(Z+H)| \geqq c''q$. Therefore, $\sum_H |N \det(Z+H)|^{-\sigma}$ is dominated by $\sum_{q \text{ integral}} q^{-\sigma + \frac{1}{2}pn(n+1)}$ which converges if $\sigma > \frac{1}{2}pn(n+1) + 1$. This is probably (almost certainly) not the best exponent of convergence, but our only aim has been to show the existence of a finite exponent of convergence.

In particular we have shown, in view of (I) and (II), that $\sum_{C,D} |N\det(CZ+D)|^{-\rho}$ converges for sufficiently large ρ, where (C,D) runs over all left non-associate pairs such that some $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma'$. Therefore $a(Z)=[\sum_{C,D}|N\det(CZ+D)|^{-\rho}]^{2/\rho}$ is what Pyateckii-Shapiro [10] calls a normal majorant function for Γ', and according to [10, p. 24 (Lemma 5)] it follows that $\dim H_{n,\Gamma'}(k)<+\infty$.[3]

3. The operator Φ and the ringed structure on $\mathfrak{S}_n^*/\Gamma'$. In this section we shall supply the space $\mathfrak{S}_n^*/\Gamma'$ with a ringed structure, i.e., a subsheaf $\mathfrak{A}$ of the sheaf of germs of continuous, complex-valued functions on $\mathfrak{S}_n^*/\Gamma'$, with respect to which it will be proved later that $\mathfrak{S}_n^*/\Gamma'$ is a normal general analytic space. Because the case $p=1$ has been dealt with in [1, 11, 12], we shall assume here that $p>1$ in order to avoid treatment of special cases.

As usual, Γ' denotes a group commensurable with $\Gamma=\Gamma_{n,p}$. We let Ω, $\mathfrak{U}$, $\mathfrak{S}_r^p$, $\mathfrak{U}_r$, $M_{r\lambda}$, $\mathfrak{S}_{r\lambda}^p$, $\mathfrak{U}_{r\lambda}$, $M_{sv}^{r\lambda}$, and $\Gamma_{r\lambda}$ have the same meanings as in § 1. Moreover, we let $\mathfrak{B}_{r\lambda}=\mathfrak{S}_{r\lambda}^p/M_{r\lambda}\Gamma_{r\lambda}M_{r\lambda}^{-1}$ and $\mathfrak{B}_n^*=\mathfrak{S}_n^*/\Gamma'$. Then $\mathfrak{B}_n^*=\bigcup_{r,\lambda}\mathfrak{B}_{r\lambda}$.

We denote by $\mathfrak{H}_{n,\Gamma',\mathfrak{U}_n}(k,v)$ the module of holomorphic functions on $\mathfrak{U}_n$ satisfying

$$f(MZ)=N\det(CZ+D)^k f(Z)v(M)^{-1},\qquad Z\in\mathfrak{U}_n,$$

for $M\in\Gamma'$, k being a non-negative integer and v having its usual meaning. Let $f\in\mathfrak{H}_{n,\Gamma',\mathfrak{U}_n}(k,v)$ and let $Z_0\in\mathfrak{U}_r$. By appropriately restricting $\mathfrak{U}$, we may asusme that $\mathfrak{U}$ is stable under all the *real* translations: $Z\to Z+T$, $Z=(Z^1,\cdots,Z^p)$, $T=(T^1,\cdots,T^p)$, $T^j=\begin{pmatrix}0 & T_{12}^{\,j}\\ {}^tT_{12}^{\,j} & T_2^{\,j}\end{pmatrix}$, ${}^tT_2^{\,j}=T_2^{\,j}$. As in [12, p. 14-02], we have

$$\Gamma'\cap\mathfrak{G}_r^n\supset\Gamma'_{Z_0}\supset\Gamma'\cap\mathfrak{N}_r^n.$$

Since $\Gamma'\cap\mathfrak{N}_r^n$ is commensurable with $\Gamma\cap\mathfrak{N}_r^n$, it follows that there exists an integer $q\in\mathfrak{o}$, $q\neq 0$, and a subgroup γ_{n-r}' of finite index in the group of

[3] Since it may not be too easy to establish the bounded convergence of the series for $a(Z)$ in the set $\Omega(\mu,t)$, a normal majorant function may be obtained easily in the following way: By using the fact that the image under $\Phi_{r\lambda}^n$ is everywhere dense in $\mathfrak{S}_{r\lambda}$ (a fact which does not make use of the finite dimensionality of the latter) one may establish the existence of a finite number of modular forms $\phi_1,\cdots,\phi_N$ of a suitable high weight having no common zeros on $\mathfrak{S}_n^*$. Then a suitable power of the quantity $|\phi_1|^2+\cdots+|\phi_N|^2$ provides us with a normal majorant function.

$(n-r)\times(n-r)$ unimodular matrices over $\mathfrak{o}$ such that $\Gamma' \cap \mathfrak{N}_r^n$ contains all of the transformations:

$$\text{(i)}\quad M=\begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1}\end{pmatrix},\ U=\begin{pmatrix} E_r & 0 \\ 0 & U_2\end{pmatrix},\ U_2\in\gamma_{n-r}'$$

$$\text{(ii)}\quad M=\begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1}\end{pmatrix},\ U=\begin{pmatrix} E_r & U_{12} \\ 0 & E_{n-r}\end{pmatrix},\ U_{12}\equiv 0 \pmod q$$

$$\text{(iii)}\quad M=\begin{pmatrix} E & T \\ 0 & E\end{pmatrix},\ T=\begin{pmatrix} 0 & T_{12} \\ {}^tT_{12} & T_2\end{pmatrix},\ T_2={}^tT_2,\quad T_{12},T_2\equiv 0 \pmod q$$

where we may suppose v to be trivial on M in (i), (ii), and (iii). Therefore f has a Fourier expansion in the connected component of $\mathcal{U}_n$ to the closure of which belongs Z_0:

$$\begin{aligned} f(Z) &= \sum_{S_2} a_{S_2}(Z_1,Z_{12})\epsilon(S_2Z_2) \\ &= \sum_{S_2,S_{12}} b_{S_2,S_{12}}(Z_1)\epsilon({}^tS_{12}Z_{12}+S_2Z_2)\end{aligned}$$

the summation being over matrices S_2, S_{12} with entries in $\mathfrak{k}$ such that $\operatorname{tr}(S_2T_2)$, $\operatorname{tr}({}^tS_{12}T_{12})$ are integral if T is as in (iii). These series converge uniformly on every compact subset of $\mathcal{U}_n$.

From the uniqueness of the Fourier expansions and from the fact that f must be invariant under a subgroup of finite index in $\Gamma' \cap \mathfrak{N}_r^n$, we see as in [12] that the coefficients a_{S_2} and $b_{S_2,S_{12}}$ must satisfy the following:

(a) $a_{S_2[{}^tU_2]}(Z_1,Z_{12}) = \det U^k a_{S_2}(Z_1,Z_{12}U_2)$,

(b) $a_{S_2}(Z_1,Z_{12}+Z_1U_{12}) = \det U^{-k} a_{S_2}(Z_1,Z_{12}) \cdot\epsilon(-S_2(Z_1[U_{12}]+2{}^tU_{12}Z_{12}))$

(c) $a_{S_2}(Z_1,Z_{12}+T_{12}) = a_{S_2}(Z_1,Z_{12})$

(d) $b_{S_2[{}^tU_2],S_{12}{}^tU_{12}}(Z_1) = \det U^k b_{S_2,S_{12}}(Z_1)$

(e) $b_{S_2,S_{12}+2U_{12}S_2}(Z_1) = \det U^k b_{S_2,S_{12}}(Z_1)\epsilon(S_2Z_1[U_{12}]+{}^tS_{12}Z_1U_{12})$

for all M in (i), (ii), (iii). We say that S_{12} is rational multiple of S_2 if there exists an $r\times(n-r)$ matrix W over $\mathfrak{k}$ such that $S_{12}=WS$. It is seen just as in [12] that if $n\geqq 2$, $b_{S_2,S_{12}}(Z_1)\not\equiv 0$ implies that S_{12} is a rational multiple of S_2. If $r>0$, we have from (d) (putting $U_{12}=qU_{12}'$):

$$\begin{aligned} &a_{S_2}(Z_1,Z_{12}) \\ &= \sum_{S_{12} \bmod \{2qU_{12}'S_2\}} \ \sum_{U'_{12} \bmod U(S_2)} b_{S_2,S_{12}+2qU_{12}'S_2}(Z_1)\epsilon({}^tS_{12}Z_{12}+2qS_2{}^tU_{12}'Z_{12}) \\ &= \sum_{S_{12}} \sum_{U_{12}'} \det U^k b_{S_2,S_{12}}(Z_1)\,\epsilon(q^2S_2Z_1[U_1'{}_2]+q{}^tS_{12}Z_1U_{12}' \\ &\qquad +2qS_2{}^tU_{12}'Z_{12}+{}^tS_{12}Z_2),\end{aligned}$$

$U(S_2)$ denoting the set of U_{12}' such that $U_{12}'S_2 = 0$. The series

$$\sum_{U_{12}'} |\epsilon(q^2 S_2 Z_1[U_{12}'] + q({}^t S_{12} Z_1 + 2S_2 {}^t Z_{12}) U_{12}')|$$

must therefore converge. There exists a non-singular matrix P over $\mathfrak{k}$ such that $S_2 = P\begin{pmatrix} S_2^0 & 0 \\ 0 & 0 \end{pmatrix}{}^t P$, $S_{12} = (S_{12}^0 0)\,{}^t P$, where S_2^0 is a non-singular matrix of rank $t \leqq n - r$ and S_{12}^0 is $r \times t$. Then the convergence of the above series of absolute values implies that of the following series:

$$\sum_{U_{12}''} |\epsilon((q_1)^2 S_2^0 Z_1[U_{12}''] + q_1({}^t S_{12}^0 Z_1 + 2S_2^0 {}^t Z_{12}^0) U_{12}''|,$$

the summation being taken over all $r \times t$ matrices over $\mathfrak{o}$. Choosing $U_{12}'' = (g\,0 \cdots 0)$, where g is any $(n-r)$-(column) vector in $\mathfrak{o}^t$, we see that

$$\sum_g \exp(-2\pi (\sum_\rho [(q_1^{(\rho)})^2 y_{11}^{(\rho)} (S_2^0)^{(\rho)} [g^{(\rho)}] + {}^t b^{(\rho)} g^{(\rho)}]))$$

must converge, where b depends on S_{12}^0, S_2^0, Z_1, Z_{12}^0, the summation on ρ being over the distinct isomorphisms of $\mathfrak{k}$ into R. Suppose S_2^0 is not positive definite. Choose g such that $(S_2^0)^{(\sigma)}[g^{(\sigma)}] < 0$ and let e be a unit in $\mathfrak{k}$ such that $|e^{(\sigma)}| > 1$, $|e^{(\rho)}| < 1$ if $\rho \neq \sigma$. Then

$$\sum_m \exp(-2\pi (\sum_{\rho \neq \sigma} y_{11}^{(\rho)} e^{(\rho) 2m} (S_2^0)^{(\rho)} [g^{(\rho)}] + y_{11}^{(\sigma)} e^{(\sigma) 2m} (S_2^0)^{(\sigma)} [g^{(\sigma)}] + \sum_\rho {}^t b^{(\rho)} g^{(\rho)} (e^{(\rho) 2m}))$$

must converge, being a subseries of the above — but this is evidently impossible. Hence $S_2^0 \gg 0$.

If $r = 0$, we have the following from (a)

$$a_{S_2[{}^t U_2]} = \det U^k a_{S_2} \tag{1}$$

as well as the estimate

$$|a_{S_2}| \leqq K \exp(\mathrm{tr} S_2) \tag{2}$$

derived from the convergence of the Fourier series $\sum a_{S_2} \epsilon(S_2 Z_2)$. From (1) and (2) we easily deduce

$$|a_{S_2}| \leqq K |\det U^k| \exp(\mathrm{tr}({}^t U S_2 U)) \tag{3}$$

for any $n \times n$ unimodular $U \equiv E \pmod q$. If $n \geqq 2$, and if $S_2^{(\sigma)}$ is indefinite, we can "isolate" the terms $\mathrm{tr}({}^t U^{(\sigma)} S_2^{(\sigma)} U^{(\sigma)})$ by finding a unit $e \equiv 1 \pmod{N(q)}$ such that $|e^{(\rho)}| < 1$ if $\rho \neq \sigma$, $|e^{(\sigma)}| > 1$, and then multiplying U by e^m for sufficiently large m. Then by Koecher's argument [12 exp. 4] it follows that

$a_{S_2} = 0$. If $n = 1$, $\operatorname{tr}({}^tUS_2U) = \sum_\rho s^{(\rho)}(u^{(\rho)})^2$. Supposing $s^{(\sigma)} < 0$, we could choose e as before and let $u^{(\sigma)} = (e^{(\sigma)})^m$. Using (3) and letting $m \to \infty$ we easily deduce that $a_S = 0$ (this is the same argument as in [6]). Therefore, if $p > 1$, the restriction $n > 1$ is unnecessary in order to guarantee that $a_{S_2} \neq 0$ only if $S_2 \geqq 0$.

Since $a_0(Z_1, Z_{12}) = b_{00}(Z_1)$ depends only on Z_1 and is a holomorphic function in a neighborhood of Z_0, we may define $f_r(Z_1) = (\Phi_r{}^n f)(Z_1) = b_{00}(Z_1)$. Then it may be verified as in [12, pp. 14-08 to 14-10] that $f_r \in \mathfrak{H}_{n,\Gamma_r',U_r}(k, v_r)$, where $\Gamma_r' = \pi_r(\Gamma' \cap \mathfrak{G}_r{}^n)$ and $v_r(M_1) = \det D_2{}^{-k} v(M)$ acts naturally on a certain quotient space of C^q ($v(M) \in GL(q, C)$) which, for our purposes, is unnecessary to specify (here

$$M = \begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & A_2 & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & D_2 \end{pmatrix} \in \Gamma' \cap \mathfrak{G}_r{}^n \cap \pi_r{}^{-1}(M_1)).$$

Moreover, it is not hard to verify that $\lim_{Z_\nu \to Z_0} f(Z_\nu) = f_r(Z_0)$ for any sequence $\{Z_\nu\} \subset \mathcal{U}_n$ converging to Z_0. Finally, if $s < r < n$, it may be verified without difficulty that $\Phi_s{}^r \Phi_r{}^n = \Phi_s{}^n$.

Let $M \in Sp(n, \mathfrak{k})$. Then as in [12, p. 14-13], $f \mid M \in \mathfrak{H}_{M^{-1}\Gamma' M, M^{-1}\mathcal{U}_n}(k, v_M)$, where $v_M(M^{-1}M'M) = v(M')$, so that

$$\Phi_r{}^n(f \mid M) \in \mathfrak{H}_{\pi_r(M^{-1}\Gamma' M \cap \mathfrak{G}_r{}^n),\, M^{-1}\mathcal{U} \cap \mathfrak{S}_r{}^p}(k, v_{Mr}).$$

If, furthermore, $L \in \mathfrak{G}_r{}^n$, it can easily be shown that $\Phi_r{}^n(f \mid ML) = \Phi_r{}^n(f \mid M) \mid \pi_r(L)$. We define

$$f_{r\lambda} = \Phi_{r\lambda}{}^n f = \Phi_r{}^n(f \mid M_{r\lambda}),$$

for $f \in \mathfrak{H}_{n,\Gamma',\mathcal{U}_n}(k, v)$, so that

$$f_{r\lambda} \in \mathfrak{H}_{r,\Gamma_{r\lambda},\mathcal{U}_{r\lambda}}(k, v_{r\lambda}).$$

If furthermore $s < r < n$ and if $(\lambda, \nu) \to \mu$, then

$$\Phi_{s\nu}{}^{r\lambda} \Phi_{r\lambda}{}^n f = \Phi_{s\mu}{}^n f \mid \pi_s(L)$$

if $M_{r\lambda} M_{s\nu}{}^{r\lambda} = M' M_{s\mu} L$, $M' \in \Gamma'$, $L \in \mathfrak{G}_s{}^n$. The proofs of the above, being purely formal, are in no way different from those in [12]. By means of the correspondence: $f \to f \circ M_{r\lambda}{}^{-1}$, the module $\mathfrak{H}_{r,\Gamma_{r\lambda},\ r\lambda}(k, v_{r\lambda})$ can then be identified with $\mathfrak{H}_{r,M_{r\lambda}\Gamma_{r\lambda}M_{r\lambda}{}^{-1},M_{r\lambda}U_{r\lambda}}(k, v_{r\lambda}{}^*)$, where $v_{r\lambda}{}^*(M_{r\lambda}M_rM_{r\lambda}{}^{-1}) = v_{r\lambda}(M_r)$, $M_r \in \Gamma_{r\lambda}$. In particular, $\mathfrak{H}_{r,\Gamma_{r\lambda}}(k)$ may be identified with $\mathfrak{H}_{r,M_{r\lambda}\Gamma_{r\lambda}M_{r\lambda}{}^{-1}}(k)$.

For fixed r, λ, let $\mathfrak{S}_{r\lambda}(k, v_{r\lambda})$ denote the intersection (in $\mathfrak{H}_{r,\Gamma_{r\lambda}}(k, v_{r\lambda})$) of the kernels of all the maps $\Phi_{sv}{}^{r\lambda}$. $\mathfrak{S}_{r\lambda}(k, v_{r\lambda})$ is known as the module of cusp forms. Let

$$\mathfrak{H}_r(k) = \prod_\lambda \mathfrak{H}_{r,\Gamma_{r\lambda}}(k, v_{r\lambda}), \qquad \mathfrak{S}_r(k) = \prod_\lambda \mathfrak{S}_{r\lambda}(k, v_{r\lambda}),$$

and in particular let $\mathfrak{H}_{0\lambda}(k) = \mathfrak{S}_{0\lambda}(k) = C$ for all λ. It is clear that the homomorphisms $\Phi_{r\lambda}{}^n$ taken together give a single homomorphism

$$\Phi_{(r)}{}^n : \mathfrak{H}_n(k) \to \mathfrak{H}_r(k).$$

In general, $\Phi_{(r)}{}^n$ is not onto, but, as we shall see, $\mathfrak{S}_r(k)$ is contained in the image of $\Phi_{(r)}{}^n$.

If $f \in \mathfrak{H}_{n,\Gamma',\mathcal{U}_n}(k)$, we consider the collection $(f_{r\lambda})$, $f_{r\lambda} = \Phi_{r\lambda}{}^n f$, all r, λ. Such a collection is called a modular form of weight k on $\mathcal{U}$. By virtue of the mappings $M_{r\lambda}$, a modular form ϕ of weight k on $\mathcal{U}$ gives rise to a modular form of weight k with respect to $M_{r\lambda}\Gamma_{r\lambda}M_{r\lambda}{}^{-1}$ on $M_{r\lambda}U_{r\lambda} \subset \mathfrak{S}_{r\lambda}{}^p$. Because of the continuity of the mapping Φ (i.e., $\lim_{Z_\nu \to Z_0} f(Z_\nu) = f_r(Z_0)$), it is easily seen that there is a certain complex line bundle $F(k)$ over $\mathfrak{V}_n{}^*$ such that the modular forms of weight k on $\mathcal{U}$ in a natural manner represent continuous cross-sections of $F(k)$ over the natural image of $\mathcal{U}$ in $\mathfrak{V}_n{}^*$, and that $F(k)$ is uniquely defined by this requirement.

If $f \in \mathfrak{H}_{n,\Gamma'}(k, v)$, f is invariant under a group $\mathfrak{T}$ of translations: $Z \to Z + T$ of finite index in the group $\Gamma_\infty{}'$ of all translations in Γ'. Therefore it is easily seen that f has a Fourier expansion

$$f(Z) = \sum a_S \epsilon(SZ), \qquad a_S \in M_q(C),$$

where the summation is extended over all S such that $\mathrm{tr}(ST)$ is an integer for all $T \in \mathfrak{T}$. If we define $(f \mid M)(Z) = N \det(CZ + D)^{-k} f(MZ)$ for $M \in Sp(n, \mathfrak{k})$, it is not hard to show that $f \mid M \in \mathfrak{H}_{n,M^{-1}\Gamma' M}(k, v_M)$, where $v_M(M^{-1}M'M) = v(M')$. $(f \mid M)$ also has a Fourier expansion $(f \mid M)(Z) = \sum a_{S_M} \epsilon(S_M Z)$. If for each M it is true that $a_{S_M} \neq 0$ implies $S_M \geqq 0$, f is called integral. If for each M it is true that $a_{S_M} \neq 0$ implies $S_M > 0$, f is clearly a cusp form (by the continuity of the operator Φ). We have seen that $p > 1$ or $n > 1$ implies that every modular form is an integral form. Satake has proved [12], in the case $p = 1$, that if $\beta k \geqq 2n$, then $\mathfrak{S}_n(k) = \mathfrak{H}_{n,\Gamma'}{}^\beta(k)$. In our case $(p > 1)$, the proof that $\mathfrak{S}_n(k) \subset \mathfrak{H}_{n,\Gamma'}{}^\beta(k)$ is exactly as on p. 7-09 of [12], because in order to show that a given $f \in \mathfrak{H}_{n,\Gamma'}(k)$ belongs to $\mathfrak{H}_{n,\Gamma'}{}^\beta(k)$, it is sufficient to show, since Ω has finite (invariant) volume, that $(f \mid M_{n-1\lambda})$ approaches zero exponentially as $y_{nn}{}^i$ approach $+\infty$ in the domain Ω for each λ; and this condition is evidently satisfied for the cusp forms (and

not merely for those such that $a_{S_M} \neq 0$ implies $S_M >> 0$; we can see this by applying the more or less obvious generalizations of Lemmas VII and VIII of [1] for $Y \in \Omega$ and observing that as a result of these the Fourier series for $f \mid M$ can be written as the sum of a finite number of terms, each of which is $\mathcal{O}(e^{-cy_{nn}})$ for some $c > 0$. However, inasmuch as consideration of only the forms for which $a_{S_M} \neq 0$ implies $S_M >> 0$ is sufficient for our purposes, we do not dwell on the details here, and only remark the above for the sake of completeness. As a matter of fact, though, we shall shortly see that the latter type of cusp form is the *only* type of cusp form.) In case $k\beta \geqq 2(n+1)$, it can be proved that $\mathfrak{H}_{n,\Gamma'}{}^{\beta}(k) \subset \mathfrak{s}_n(k)$, in a manner analogous to that on pp. 9-33 to 9-35 of [12] by considering each of the finitely many $\delta(n-1)$-dimensional vertices of $B'(\Omega)$ in turn, noting that $M^{-1}\Gamma' M \cap \mathfrak{G}_n{}^{n-1}$ is commensurable with $\Gamma \cap \mathfrak{G}_n{}^{n-1}$ for $M \in Sp(n, \mathfrak{k})$. In fact, using the same line of reasoning as on pp. 9-33 to 9-35 of [12] it is not hard to see that if $Z_1 \in \mathfrak{S}_{n-1}{}^{p}$, $(\Phi_{n-1}{}^{n} f)(Z_1) \neq 0$, and if $V^{(n)}(\mathcal{U}_{n-1}, K)$ is a suitably small neighborhood of Z_1 in $\mathfrak{S}_n{}^*$, then

$$\int_{V^{(n)}(\mathcal{U}_{n-1}, K)} \| N \det(Y)^{k/2} f(Z) \|^{\beta} \, dZ$$

is bounded from below by

$$c \cdot \int_K^{\infty} \int_K^{t d_n{}^{(p)}} \cdots \int_K^{t d_n{}^{(2)}} (d_n{}^{(1)} \cdots d_n{}^{(p)})^{\frac{1}{2}k\beta - n - 1} \, d d_n{}^{(1)} \cdots d d_n{}^{(p)}$$

$$\geqq c' \int_K^{\infty} (d_n{}^{(p)})^{p(\frac{1}{2}k\beta - n - 1) + p - 1} \, d d_n{}^{(p)}$$

$(c, c' > 0)$ which diverges if $p(\frac{1}{2}k\beta - n - 1) + p \geqq 0$, i.e., if $\frac{1}{2}k\beta \geqq n$ or $k\beta \geqq 2n$. This characterization of the cusp forms makes it clear that if $f \in \mathfrak{s}_{n,\Gamma'}(k)$ and if $M \in Sp(n, \mathfrak{k})$, then $(f \mid M) \in \mathfrak{s}_n(k)$, because if $|N \det Y|^{k\beta/2} |f(Z)|^{\beta}$ is integrable over $B'(\Omega)$, then $|N \det Y|^{k\beta/2} |(f \mid M)|^{\beta}$ is integrable over $M^{-1}B'(\Omega)$ which is a fundamental open set for $M^{-1}\Gamma' M$. On the other hand, if the Fourier expansion of f contains the term $a \cdot \epsilon(SZ)$, $a \neq 0$, that of $f \mid M$ contains the term $ca \cdot \epsilon(S[{}^tU]Z)$ if $M = \begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix}$, where $c = N(\det U)^{-k}$, and by appropriate choice of U we may assume $S[{}^tU] = \begin{pmatrix} S^* & 0 \\ 0 & 0 \end{pmatrix}$, where $S^* >> 0$ has the same rank as S. If rank $S^* < n$, it is clear that $\Phi_{n-1}{}^{n}(f \mid M) \neq 0$, which is a contradiction. Hence the cusp forms f are just those such that if $f \mid M = \sum a_{S_M} \epsilon(S_M Z)$, $M \in Sp(n, \mathfrak{k})$, then $a_{S_M} \neq 0$ implies $S_M >> 0$. Finally we observe that for $S >> 0$ we have $E_{\Gamma';S} \in \mathfrak{H}_{\Gamma'}{}^{1}(k) = \mathfrak{s}_n(k)$ if k is sufficiently large.

As we have seen, the modular forms of a given type form a finite dimensional vector space over C, and in particular the cusp forms are a (finite-dimensional) subspace of this. If as in [12] we define

$$H_k(S) = \int_{Y>>0} N\det Y^k \exp(-\pi\mathrm{tr}(SY))N\det Y^{-n-1}\,dY$$

$$= \int_{Y>>0} N\det Y^{k-n-1}\exp(-\pi\mathrm{tr}(SY))\,dY,$$

we may verify by the same steps as in [12, pp. 9-08 to 9-09] or [9, pp. 574-5] that if $f\in\mathfrak{S}_n(k,v)$ and if $S>>0$, ω given, then

$$\langle f, E_{S;\omega}\rangle = c\cdot\mathrm{tr}[\omega^* H_k(4S)f(S)], \tag{4}$$

* denoting conjugate transpose, c a constant depending on $\mathfrak{k}$ and S and $\hat{f}(S)$ the coefficient of $\epsilon(SZ)$ in the Fourier expansion of f, and where $\langle f,g\rangle$ is defined for $f,g\in\mathfrak{H}_{n,\Gamma'^2}(k,v)$ by

$$\langle f,g\rangle = \int_{Z \bmod \Gamma} \mathrm{tr}(g(Z)^* N\det Y^k f(Z))\,dZ, \tag{5}$$

dZ being the $Sp(n,\mathfrak{k})$-invariant measure on $\mathfrak{S}_n^p$.

In view of (4) it is clear that the Eisenstein series $E_{S;\omega}$ span the space of cusp forms for Γ. By the use of the same "cutting down" and "averaging" processes as on pp. 16-08 to 16-11 of [12], it is not difficult to see that this result may be generalized from Γ to Γ' and that $\mathfrak{o}_{n,\Gamma'}(k,v)$ is spanned by the transforms of the Eisenstein series $E_{\Gamma';S,\omega}$, $S>>0$, by $M\in Sp(n,\mathfrak{k})$ (i.e., by the sedies $E_{M^{-1}\Gamma' M;S,\omega}\,|\,M^{-1}$, $S>>0$, $M\in Sp(n,\mathfrak{k})$).

4. Global ontoness of Φ. We refer here to the notation of § 3 and the approach of [12, pp. 16-01 ff.]. Our aim is to show for each r, $0\leqq r\leqq n-1$, that $\mathfrak{o}_r(k)\subset\Phi_{(r)}{}^n(\mathfrak{H}_n(k))$. For this it is clearly sufficient to show that if $f_{r\lambda}\in\mathfrak{o}_{r\lambda}(k)$, then there exists $f\in\mathfrak{H}_n(k)$ such that $\Phi_{r\lambda}{}^n f = f_{r\lambda}$, $\Phi_{r\lambda'}{}^n f = 0$ if $\lambda'\neq\lambda$. Since $\mathfrak{o}_{r\lambda}(k)$ is generated by the transforms of the series $E_{\Gamma_r\lambda';S_r}$, $S_r>>0$, it follows that it is sufficient to establish for each $S_r>>0$ and $L_1\in Sp(r,\mathfrak{k})$ the existence of $f\in\mathfrak{H}_n(k)$ such that

$$\Phi_{(r)}{}^n f = (0,\cdots,0,E_{L_1\Gamma_r\lambda' L_1^{-1};S_r}\,|\,L_1,0,\cdots,0).$$

We shall prove this by computing $\Phi_{(r)}{}^n(E_{M_0^{-1}\Gamma' M_0;S}\,|\,M_0^{-1})$, $M_0\in Sp(n,\mathfrak{k})$. If

$$M_0 = \begin{pmatrix} {}^tU_0 & T_0U_0^{-1} \\ 0 & U_0^{-1}\end{pmatrix}\in\mathfrak{G}_0{}^n, \qquad E_{M_0\Gamma' M_0^{-1};S_i}\,|\,M_0 = (\det U_0)^k\epsilon(ST_0)E_{\Gamma';U_0S^tU_0}.$$

Therefore, we may assume $S=\begin{pmatrix} S_0 & 0\\ 0 & 0\end{pmatrix}$, $S_0>>0$ of rank t.

Because the class number of $\mathfrak{k}$ is in general >1, we cannot in general say that every element M of $\Gamma_{n,p}$ can be written in the form

$$M=\begin{pmatrix} {}^tU & TU^{-1} \\ 0 & U^{-1} \end{pmatrix}\begin{pmatrix} A_0 & 0 & B_0 & 0 \\ 0 & E & 0 & 0 \\ C_0 & 0 & D_0 & 0 \\ 0 & 0 & 0 & E \end{pmatrix}\begin{pmatrix} {}^tV & 0 \\ 0 & V^{-1} \end{pmatrix},$$

where $\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \Gamma_{s,p}$, $\det C_0 \neq 0$, ${}^tT = T$ has coefficients in $\mathfrak{o}$, and U, V belong to the group of $n \times n$ unimodular matrices over $\mathfrak{o}$ (as is the case, according to Siegel [14], if the class number of $\mathfrak{k}$ is 1). However, every $M \in Sp(n,\mathfrak{k})$ can be written in the above form if we require only that $U, V \in GL(n,\mathfrak{k})$, and $\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in Sp(s,\mathfrak{k})$, $\det C_0 \neq 0$. Let $Z_0 \in \mathfrak{S}_r^p$. We now wish to compute $\lim_{Z \to Z_0} \epsilon(SZ)\,|M$, $S=\begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix}$, $S_0 >> 0$ of rank t, $M \in Sp(n,\mathfrak{k})$. If $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathfrak{k})$ is written in the above form, we have

$$\begin{aligned} |N\det U|\,|N\det(CZ+D)| &= |N\det C_0|\,|N\det(Z[Q]+C_0^{-1}D_0)| \\ &\geqq |N\det C_0|\,N\det Y[Q], \end{aligned}$$

where Q is the $n \times s$ matrix made up of the first s-columns of V. By replacing V by VU^*A_μ, where A_μ is one of a *finite* number of matrices in $GL(n,\mathfrak{k})$ and U^* is unimodular, $A_\mu=\begin{pmatrix} A_\mu' & 0 \\ 0 & E \end{pmatrix}$, $U^*=\begin{pmatrix} U^{*\prime} & 0 \\ 0 & E \end{pmatrix}$, we may assume $Y[Q] \in Q(t)$ and since the fixed C_0 is then replaced by $C_0{}^tA_\mu'^{-1}U^{*\prime-1}$, we may assume that $|N\det C_0| \geqq c_1$. If $Y[Q] \in Q(t)$,

$$|N\det Y[Q]| \geqq c_2 \prod_{i=1}^{s}\prod_{j=1}^{p} Y^j[q_i^j],$$

where c_2 is a constant (depending only on n and s) and where q_i are the columns of Q. Therefore

$$\lim_{Z \to Z_0} \epsilon(SZ)\,|M = \lim_{Z \to Z_0} \epsilon(S \cdot MZ)N\det(CZ+D)^{-k} = 0$$

unless $Q=\begin{pmatrix} Q_1 \\ 0 \end{pmatrix}$, Q_1 being $r \times s$, $s \leqq r$. In the latter case we may assume

$$V=\begin{pmatrix} V_1 & 0 \\ 0 & E \end{pmatrix},$$

V_1 being $r \times r$. Proceeding further as in [12, pp. 16-13 to 16-14] we deduce

further that $\lim_{Z\to Z_0} \epsilon(SZ)\,|\,M = 0$ unless the matrix P composed of the first r columns of U is of the form

$$P = \begin{pmatrix} P_1 \\ 0 \end{pmatrix}$$

P_1 being $r \times r$, and finally unless $M = NL$, $N \in \mathfrak{G}_S$, $L \in \mathfrak{G}_r^n$, so that if

$$L = \begin{bmatrix} A_1 & 0 & B_1 & * \\ * & A_2 & * & * \\ C_1 & 0 & D_1 & * \\ 0 & 0 & 0 & D_2 \end{bmatrix}, \quad L_1 = \pi(L) = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in Sp(r, \mathfrak{k}),$$

we have $\lim_{Z\to Z_0} \epsilon(SZ)\,|\,M = N(\det U)^{-k} N(\det D_2)^{-k} \epsilon(S_0 Z_0)\,|\,L_1$. For given M, it is clear that $(\det U)(\det D_2)^{-1}$ depends only on L_1.

Then if we wish to calculate

$$\Phi_{r\lambda}{}^n(E_{M_0{}^{-1}\Gamma' M_0;S}\,|\,M_0{}^{-1}) = \Phi_r{}^n(E_{M_0{}^{-1}\Gamma' M_0;S}\,|\,M_0{}^{-1}M_{r\lambda}),$$

it is evidently sufficient to calculate

$$\Phi_r{}^n(E_{\Gamma'';S}\,|\,M_1)(Z_0) = \lim_{Z\to Z_0} \sum_{M:\,\Gamma_S''\backslash\Gamma'' M_1} \epsilon(SZ)\,|\,M,$$

where we may suppose that Z remains in the union of a finite number of translates of Ω under elements of $Sp(n, \mathfrak{k})$. Since the calculations are purely formal in nature, apart from establishing the region of uniform convergence of a simple Eisenstein series in one variable, we omit these with the remark that the computations are just as in [12, pp. 16-04 to 16-06 and pp. 16-15 to 16-16; on line 6 from the top of p. 16-16, $C_{M_4} = 0$ should be $C_{M_4} \neq 0$] and write down the final result that if $S = \begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix}$, where $S_0 >> 0$, $\operatorname{rank} S_0 = r$, then

$$\Phi_r{}^n(E_{\Gamma'';S}\,|\,M_1) = \begin{cases} cE_{\pi_r(\Gamma''\cap\mathfrak{G}_r{}^n);S_0}\,|\,L_1 \text{ if } \Gamma'' M_1 \cap \mathfrak{G}_r{}^n \neq \emptyset, \\ (L \in \Gamma'' M_1 \cap \mathfrak{G}_r{}^n,\ L_1 = \pi_r(L)), \\ 0 \text{ otherwise}, \end{cases}$$

where $c \neq 0$.

Therefore, letting $\Gamma'' = M_0{}^{-1}\Gamma' M_0$, $M_1 = M_0{}^{-1}M_{r\lambda}$,

$$\Phi_{r\lambda}{}^n(E_{M_0{}^{-1}\Gamma' M_0;S}\,|\,M_0{}^{-1}) = \begin{cases} cE_{L_1\Gamma_{r\lambda}L_1{}^{-1};S_0}\,|\,L_1 \text{ if } M_0 \in \Gamma' M_{r\lambda}\mathfrak{G}_r{}^n, \\ (L \in M_0{}^{-1}\Gamma' M_{r\lambda} \cap \mathfrak{G}_r{}^n,\ L_1 = \pi_r(L),\ c \neq 0), \\ 0 \text{ otherwise}. \end{cases}$$

Since, as remarked, the space of cusp form for $\Gamma_{r\lambda}'$ is generated by the transforms $E_{L_1\Gamma_{r\lambda}L_1{}^{-1};S_0}\,|\,L_1$ of the series $E_{\Gamma_{r\lambda}';S_0}$, $S_0 >> 0$, we see that $\mathfrak{d}_r \subset \Phi_{(r)}{}^n\mathfrak{H}_n$.

5. The analytic structure on $\mathfrak{S}_n^*/\Gamma_{n,p}'$. The compactified space $\mathfrak{S}_n^*/\Gamma_{n,p}'$ can be written as a union:

$$\mathfrak{V}_n^* = \mathfrak{S}_n^*/\Gamma_{n,p}' = \bigcup_r \bigcup_\lambda (\mathfrak{S}_r^p)/\Gamma_{r\lambda,p}' = \bigcup_{r,\lambda} V_{r\lambda}.$$

$V_n = \mathfrak{S}_n^p/\Gamma_{n,p}'$ is everywhere dense in this space, and for each r, λ, $V_{r\lambda} = (\mathfrak{S}_r^p)/\Gamma_{r\lambda,p}'$ is a general analytic space of dimension $\frac{1}{2}pr(r+1)$. If $\mathcal{O}$ is an open subset of $\mathfrak{S}_n^*/\Gamma_{n,p}'$ and f a complex-valued continuous function on $\mathcal{O}$, we say that f is an $\mathfrak{A}$-function on $\mathcal{O}$ if $f \mid (\mathfrak{V}_n \cap \mathcal{O})$ is analytic in the natural analytic structure on $\mathfrak{V}_n \cap \mathcal{O}$.

As we have seen, the cusp forms on $\mathfrak{S}_r^p$ of weight k with respect to $\Gamma_{r\lambda}$ can be characterized, if $k > 2n$, as precisely constituting $\mathfrak{H}_{\Gamma_{r\lambda}'}{}^1(k)$, and just as in [12, p. 10-20] this space is just the space of Poincaré series of a certain weight. By the existence theorem of [12, p. 10-44] it is known for given m that if $a_1, \cdots, a_l$ are pairwise incongruent points of $\mathfrak{S}_r^p \bmod \Gamma_{r\lambda}'$ and if k is a sufficiently large multiple of an integer k_0 depending only on $\Gamma_{r\lambda}'$, then the Poincaré series of weight k span the direct sum of the residue classes of (invariant) power series modulo the m-th power of the maximal ideal (of invariant power series) at $a_1, \cdots, a_l$. From these facts we have the following

THEOREM 2. *Let $a_1, \cdots, a_l \in \mathfrak{S}_r^p$ be pairwise incongruent* mod $\Gamma_{r\lambda}'$ *and let m be a positive integer. Then if k is a sufficiently large multiple of k_0, there exists $f \in \mathfrak{H}_{\Gamma_{r\lambda}'}(k)$ having preassigned (invariant) power series developments at $a_1, \cdots, a_l$ up to and including terms of degree $m-1$.*

Since $\mathfrak{H}_r \subset \Phi_{(r)}{}^n(\mathfrak{H}_n)$, we have the obvious

COROLLARY. *Let $x \in \mathfrak{V}_{r\lambda}$. Then if k is a sufficiently large multiple of k_0, there exist $f_0, \cdots, f_q \in \mathfrak{H}_n$ such that*

(1) $(\Phi_{r\lambda}{}^n f_0)(x) \neq 0$.

(2) *x is an isolated point of the variety of common zeros of*

$$\Phi_{r\lambda}{}^n f_1, \cdots, \Phi_{r\lambda}{}^n f_q.$$

(3) $\Phi_{(r+1)}{}^n f_j = 0$, $j = 0, \cdots, q$, *and* $\Phi_{r\lambda'}{}^n f_j = 0$, $j = 0, \cdots, q$ if $\lambda' \neq \lambda$.

Let $x \in \mathfrak{V}_{r\lambda}$, let $f_0, \cdots, f_s$ be chosen as in the above Corollary, and let $\mathcal{U}$ be a compact neighborhood of x on $\mathfrak{V}_n^*$ such that $f_0 \neq 0$ on $\mathcal{U}$. If $r \leqq r' \leqq n$, let $\mathcal{U}_{r'} = \bigcup_{\lambda'} \mathfrak{V}_{r'\lambda'} \cap \mathcal{U}$.

If $f_1, \cdots, f_k$ are functions on an open set $\mathcal{O}$, we let

$$V(f_1, \cdots, f_k) = \{z \mid z \in \mathcal{O}, f_1(z) = \cdots = f_k(z) = 0\}.$$

THEOREM 3. *The $\mathfrak{A}$-functions on $\mathcal{U}$ separate the points of $\mathcal{U}$. For each*

$s, \nu, n \geqq s \geqq r$, there exists a finite number of $\mathfrak{A}$-functions $g_1, \cdots, g_t$ on $\mathcal{U}$ such that $V(g_1, \cdots, g_t) = \operatorname{clos}(\mathfrak{V}_{s\nu} \cap \mathcal{U})$.

Proof. We prove the last part of the theorem first by induction on $n - s$. This part is trivial if $s = n$. Suppose it has been proved for $s+1$. Let ν' be such that $\operatorname{clos}(\mathfrak{V}_{s\nu}) \cap \mathcal{U} \subset \operatorname{clos}(\mathfrak{V}_{s+1\,\nu'}) \cap \mathcal{U}$, and let $g_1, \cdots, g_{t_1}$ be $\mathfrak{A}$-functions on $\mathcal{U}$ such that $V(g_1, \cdots, g_{t_1}) = \operatorname{clos}(\mathfrak{V}_{s+1\,\nu'}) \cap \mathcal{U}$. If $y \in \operatorname{clos}(\mathfrak{V}_{s+1\,\nu'}) \cap \mathcal{U} - \operatorname{clos}(\mathfrak{V}_{s\nu})$, there exists an $\mathfrak{A}$-function g on $\mathcal{U}$ such that $\mathfrak{V}_{s\nu} \subset V(g)$, $g(y) \neq 0$. In fact, either $y \in \mathfrak{V}_{s+1,\nu'}$, in which case there exists $h \in \mathfrak{H}_{s+1\,\nu'}(l)$, for some sufficiently large multiple l of k_0, such that $h(y) \neq 0$ and there exists $f \in \mathfrak{H}_n$ such that $\Phi_{s+1\,\nu'}{}^n f = h$, or else $y \in \mathfrak{V}_{s,\nu''}$, $\nu'' \neq \nu$, and there exists $f \in \mathfrak{H}_n$ such that $\Phi_{s\nu}{}^n \equiv 0$, $(\Phi_{s\nu''}{}^n f)(y) \neq 0$; in either case some power of f divided by a suitable power of f_0 is the desired $\mathfrak{A}$-function g. If $g_1, \cdots, g_{t_1}, g_{t_1+1}, \cdots, g_t$ are $\mathfrak{A}$-functions on $\mathcal{U}$ such that $\mathfrak{V}_{s\nu} \subset V(g_1, \cdots, g_t)$, $V(g_1, \cdots, g_t) - \mathfrak{V}_{s\nu}$ consists of a certain countable number of irreducible analytic spaces. Let $g_{t_1+1}, \cdots, g_t$ be chosen such that the highest dimension of any of these is as small as possible. We assert that then $V(g_1, \cdots, g_t) = \mathfrak{V}_{s\nu}$. In fact, suppose this were not the case and let $V_1, V_2, \cdots$, be all the components of $V(g_1, \cdots, g_t) - \mathfrak{V}_{s\nu}$ of highest dimension $= d$, and let $p_i \in V_i$, $i = 1, 2, \cdots$. We know there exists an $\mathfrak{A}$-function $g^{(i)}$ in $\mathcal{U}$ such that $g^{(i)}(p_i) \neq 0$, $\mathfrak{V}_{s\nu} \subset V(g^{(i)})$. Then (just as in [12]) we can construct a uniformly convergent series $\sum c_i g^{(i)} = g$, c_i constants, such that $g(p_i) \neq 0$, $i = 1, 2, \cdots$. Since the dimension of the highest dimensional component of $V(g_1, \cdots, g_t) - \mathfrak{V}_{s\nu}$ must now be strictly less than d, we have arrived at a contradiction. This completes the proof of the last part of the theorem.

Now let $y_1, y_2 \in \mathcal{U}$. Suppose $y_1 \in \mathfrak{V}_{r_1\lambda_1}$, $y_2 \in \mathfrak{V}_{r_2\lambda_2}$. It is an immediate consequence of the corollary to the preceding theorem that if $r_1 > r_2$ or if $r_1 = r_2$, $\lambda_1 \neq \lambda_2$, then for suitable l there exists $f \in \mathfrak{H}_n(l)$ such that $\Phi_{r_1\lambda_1}{}^n f(y_1) \neq 0$, $\Phi_{r_2\lambda_2}{}^n f(y_2) = 0$. If $y_1, y_2 \in \mathfrak{V}_{r_1\lambda_1}$, then there exists $h \in S_{r_1\lambda_1}(l)$ such that $h(y_1) \neq 0$, $h(y_2) = 0$, and there exists $f \in \mathfrak{H}_n$ such that $\Phi_{r_1\lambda_1}{}^n f = h$. In either case, a suitable power of f divided by some power of f_0 is an $\mathfrak{A}$-function g on $\mathcal{U}$ such that $g(y_1) \neq g(y_2)$.

COROLLARY. *There exists a finite set of $\mathfrak{A}$-functions $g_1, \cdots, g_N$ on $\mathcal{U}$ such that for suitable integers $1 < l_{n-1} < l_{n-2} < \cdots < l_r < l_0 < N$ we have $V(g_1, \cdots, g_{l_s}) = \mathcal{U}_s$, $n - 1 \leqq s \leqq r$, $V(g_1, \cdots, g_{l_0}) = x$.*

Now consider the mapping

$$g : \mathcal{U} \to C^N$$

defined by $g(y) = (g_1(y), \cdots, g_N(y))$. Since $g^{-1}g(x) = x$, this is evidently

a proper mapping if $\mathcal{U}$ is chosen small enough, and by the preceding Corollary we may assume the images $g(\mathfrak{V}_{sv})$, $n \geqq s \geqq r$, are disjoint. Proceeding just as in [1] we may assume, by suitably choosing open neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of x such that $\bar{\mathcal{U}}_2 \subset \mathcal{U}_1$, $\bar{\mathcal{U}}_1 \subset \mathcal{U}$ we may assume that $g^{-1}(g(\bar{\mathcal{U}}_2)) \cap \bar{\mathcal{U}}_1 \subset \mathcal{U}_1$ and that if $a \in g(\mathfrak{V}_{sv} \cap \bar{\mathcal{U}}_2)$, then $g^{-1}(a) \cap \mathcal{U}$ is a compact analytic subvariety of $\mathcal{U}_1 \cap \mathfrak{V}_{sv}$, and since the $\mathfrak{A}$-function on $\mathcal{U}$ separate points, $g^{-1}(a)$ is a finite set of points [1]. Therefore, $g(\mathfrak{V}_{sv} \cap \mathcal{U}_2)$ is an analytic variety of dimension $\frac{1}{2}ps(s+1)$ in a neighborhood of each of its points. It therefore follows from a theorem of Remmert and Stein [15] by an obvious procedure of induction that $g(\mathcal{U}_2)$ is an analytic variety of dimension $\frac{1}{2}pn(n+1)$ in a neighborhood $\mathcal{N}$ of $g(x)$, and the inverse image of each of an everywhere dense set of points of $g(\mathcal{U}_2) \cap \mathcal{N}$ consists of exactly d points for some positive integer d. Then, just as in [1], we can make $d = 1$ by an appropriate choice of $g_1, \cdots, g_N$, because the $\mathfrak{A}$-functions on $\mathcal{U}$ separate points. On the other hand, each point y of $\mathcal{U}$ has a basis of neighborhoods $\{\mathcal{U}_\alpha\}$ such that $\mathcal{U}_\alpha \cap \mathcal{U}_n$ is an irreducible analytic variety, and by our definition of $\mathfrak{A}$-function it follows easily [12 exposé 15] that the ring $\mathfrak{A}_y$ of germs of $\mathfrak{A}$-functions at y is integrally closed. Moreover, it follows for the same reasons as in the place cited that $g(\mathcal{U}_2)$ is irreducible at $g(x)$. If $y_1, y_2 \in \mathcal{U}_2$ are such that $g(y_1) = g(y_2)$, and if $\mathcal{V}_1$ and $\mathcal{V}_2$ are small connected neighborhoods of y_1 and y_2 respectively, it is clear that $g(\mathcal{V}_1)$ and $g(\mathcal{V}_2)$ define distinct irreducible germs of varieties at $g(y_1) = g(y_2)$. Therefore, there is a natural homeomorphism of $\mathcal{U}_2 \cap g^{-1}(\mathcal{N})$ with the canonical normal model of $g(\mathcal{U}_2) \cap \mathcal{N}$ which preserves the ringed structure [1]. Hence, provided with the given ringed structure of $\mathfrak{A}$-functions, $\mathcal{U}_2$ is a normal general analytic space. It follows at once that $\mathfrak{V}_n^*$ is a normal, general analytic space.

Let k be a multiple of k_0 and let $\phi_0, \cdots, \phi_{N(k)}$ be a basis of the modular forms of weight k. We let $V(k)$ denote the variety of common zeros of $\phi_0, \cdots, \phi_{N(k)}$. Then we can find an increasing sequence of integers $k_0 \mid k_1 \mid \cdots \mid k_n \mid k_{n+1} \mid \cdots$ such that $V(k_i) \supset V(k_{i+1})$, $i = 0, 1, \cdots$. Since $\mathfrak{V}_n^*$ is a compact general analytic space, there exists n_0 such that $V(k_{n_0}) = V(k_{n_0+1}) = \cdots$. From the Corollary of Theorem 2 it follows that if $x \in \mathfrak{V}_n^*$ and if k is a sufficiently large multiple of k_0, then there exists a modular form ϕ of weight k such that $\phi(x) \neq 0$. Therefore $V(k_{n_0})$ is empty. We denote k_{n_0} again by k_0 and in what follows assume that k is always a multiple of k_0. Since $V(\phi_0, \cdots, \phi_{N(k)})$ is empty, we may define a mapping $\Psi_k : \mathfrak{V}_n^* \to CP^{N(k)}$, where $CP^{N(k)}$ is the complex projective space of dimension $N(k)$, by $\Psi_k(x) = (\phi_0(x), \cdots, \phi_{N(k)}(x))$. Let

$$\theta_k : \mathfrak{V}_n^* \times \mathfrak{V}_n^* \to CP^{N(k)} \times CP^{N(k)}$$

be defined by $\theta_k = \Psi_k \times \Psi_k$. Let Δ_k be the diagonal of $CP^{N(k)} \times CP^{N(k)}$ and let $D_k = \theta_k^{-1}(\Delta_k)$. D_k is a subvariety of $\mathfrak{V}_n^* \times \mathfrak{V}_n^*$. If $k_1, k_2, \cdots$ is a strictly increasing sequence of positive integers such that $k_0 \mid k_1 \mid k_2 \cdots$, it is clear that $D_{k_1} \supseteq D_{k_2} \supseteq \cdots$, so that for sufficiently large N we have $D_{k_N} = D_{k_{N+1}} = D_{k_{N+2}} = \cdots$. If $x, y \in \mathfrak{V}_{n,p}^*$, there is a modular form ϕ of sufficiently high weight k such that $\phi(x) = 0$, $\phi(y) \neq 0$. This makes it clear that if $\mathfrak{D}$ is the diagonal of $\mathfrak{V}_{n,p}^* \times \mathfrak{V}_{n,p}^*$, we must have $D_{k_n} = \mathfrak{D}$, which means that Ψ_{k_N} is one-to-one.

The holomorphic functions on $\mathfrak{S}_n^p$ form an integrally closed domain of integrity. Therefore [12, p. 17-10] it follows that the integral closure A' of the graded ring A generated by $\phi_0, \cdots, \phi_{N(k)}$ (the elements of degree d are the homogeneous polynomials of degree d in $\phi_0, \cdots, \phi_{N(k)}$) is contained in the set of modular forms of weights which are multiples of k, and is finitely generated (as an algebra over C) since A' is of finite type over A. We know [12, p. 17-05, prop. 4] that there exists an integer d' such that if $A'(d')$ is the graded ring in which the elments of degree h are the automorphic forms in A' of weight hd', then $A'(d')$ is integrally closed and is generated by its elements of degree 1. Letting $g_0, \cdots, g_M$ be a basis of these elements of degree 1, the mapping

$$g: x \to (g_0(x), \cdots, g_n(x))$$

defines a 1-1 holomorphic mapping of $\mathfrak{V}_n^*$ into CP^M. Since $g^{-1}g(a) = a$, the image $g(\mathfrak{V}_n^*)$ is an analytic and therefore (by Chow's theorem) an algebraic variety. Since $A'(d')$ is (naturally identifiable with) the quotient of the algebra of homogeneous polynomials in CP^M by the homogeneous ideal of homogeneous polynomials vanishing on $g(\mathfrak{V}_n^*)$, it follows that $g(\mathfrak{V}_n^*)$ is algebraically and therefore analytically normal. Therefore g, being a one-to-one analytic mapping, is actually an isomorphism of $\mathfrak{V}_n^*$ onto $g(\mathfrak{V}_n^*)$. This completes the proof of

THEOREM 4. *$\mathfrak{V}_n^*$ is complex analytically isomorphic to a normal complex projective variety.*

Since $\dim(\mathfrak{V}_n^* - \mathfrak{V}_n) < \dim(\mathfrak{V}_n^*) - 1$ if $p > 1$, we have, just as in [1, p. 363],

THEOREM 5. *Every meromorphic function on $\mathfrak{S}_n^p$ invariant under Γ' is a quotient of modular forms. (Provided $n > 1$ or $p > 1$.)*

The referee has kindly pointed out to us a recent note [16] in which Pyateckii-Shapiro announces, without proofs, results containing most of the important results of this paper as special cases.

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THE LEBESGUE CONTANTS FOR JACOBI SERIES, II.*

By LEE LORCH.[1]

1. Introduction and statement of results. This is a direct continuation of Part I [5] whose Introduction outlines the background of this Part as well. The main results presented here are for a narrower range of the parameter α, but, for this range, are substantially more precise. The methods here are different and do not use the result established in Part I.

As before, the Lebesgue constants for Jacobi series (*i.e.*, developments in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ at the end-point $x=1$), are written, following H. Rau [6, p. 249, (40)], as

$$L_n(\alpha,\beta) = (\Gamma(n+\alpha+\beta+2)/[\Gamma(\alpha+1)\Gamma(n+\beta+1)]) \cdot \int_0^\pi (\sin\tfrac{1}{2}\theta)^{2\alpha+1}(\cos\tfrac{1}{2}\theta)^{2\beta+1} \mid P_n^{(\alpha+1,\beta)}(\cos\theta)\mid d\theta. \tag{1}$$

An asymptotic expression for $L_n(\alpha,\beta)$ is established in Part I [5, (6)] for $\alpha > -\frac{1}{2}$ and $\beta > -1$, a restriction on β which is imposed throughout this Part as well.

For $\alpha = -\frac{1}{2}$ and $-1 < \alpha < -\frac{1}{2}$ the respective results, due to G. Szegö [9, § 20], are recorded without proof in [6] and repeated in [5, (4) and (5)].

Here, [5, (6)] is sharpened for the restricted range $-\frac{1}{2} < \alpha < \frac{1}{2}$, $\alpha - \beta < 1$, by replacing the O-terms found there by an explicitly determined constant plus O-terms which are $o(1)$, the detailed statement given by (2). The same is done for [5, (4)], dealing with $\alpha = -\frac{1}{2}$, in (7).

For the first case mentioned, we have

$$L_n(\alpha,\beta) = A_{\alpha\beta}n^{\alpha+\frac{1}{2}} + B_\alpha + O(n^{\alpha-\frac{1}{2}}) + O(n^{\alpha-\beta-1}), \tag{2}$$

for $-\frac{1}{2} < \alpha < \frac{1}{2}$ *and* $\alpha - \beta < 1$. *Here*

$$A_{\alpha\beta} = 2\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\beta+\tfrac{3}{4})/\{\pi^{\frac{1}{2}}\Gamma(\alpha+1)\Gamma(\tfrac{1}{2}[\alpha+\beta]+1)\}, \tag{3}$$

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and

$$B_\alpha = \{2^{-\alpha}/\Gamma(\alpha+1)\}\{M_1(\alpha) + \int_0^{j_1} x^\alpha J_{\alpha+1}(x)\,dx$$

$$+ 2\alpha \sum_{k=1}^{\infty} (-1)^k \int_{j_k}^{j_{k+1}} x^{\alpha-1} J_\alpha(x)\,dx \tag{4}$$

$$+ 2 \sum_{k=1}^{\infty} [M_{k+1}(\alpha) - (2^{\frac{1}{2}}/\pi^{\frac{1}{2}}) \int_{j_{k-1}}^{j_k} x^{\alpha-\frac{1}{2}}\,dx]\},$$

where both infinite series are absolutely convergent.

In (4), $J_\nu(x)$ is the Bessel function of first kind and ν-th order, $j_n \equiv j_{\alpha+1,n}$ the n-th positive zero[2] of $J_{\alpha+1}(x)$, $n = 1, 2, \cdots, j_0 = 0$, and

$$M_k(\alpha) \equiv (-1)^k j_{\alpha+1,k}{}^\alpha J_\alpha(j_{\alpha+1,k}) > 0, \qquad k = 1, 2, \cdots. \tag{5}$$

$M_k(\alpha)$ is positive, as stated, since the zeros of $J_\alpha(x)$ and $J_{\alpha+1}(x)$ are interlaced [11, p. 479], *i.e.*, $j_{\alpha 1} < j_{\alpha+1,1} < j_{\alpha 2} < \cdots$, and $(-1)^k J_\alpha(j_{\alpha+1,k}) > 0$.

It is noteworthy that the constant term (4) in (2) is independent of β.

In the important particular case of Laplace series (*i.e.*, developments in terms of Legendre polynomials at the endpoint $x = 1$), $\alpha = \beta = 0$, and (2) becomes

$$L_n(0,0)$$

$$= 2(2/\pi)^{\frac{1}{2}} n^{\frac{1}{2}} + 1 + 2 \sum_{k=1}^{\infty} \{M_k(0) - (2^{\frac{1}{2}}/\pi)\,(k^{\frac{1}{2}} - [k-1]^{\frac{1}{2}})\} \tag{6}[3]$$

$$+ O(n^{-\frac{1}{2}}).$$

Here $M_k(0)$ can be interpreted as the ordinate of the k-th positive extremum (turning-point) of $|J_0(x)|$, inasmuch as the extrema of $J_0(x)$ occur at the zeros of $J_1(x)$ since $J_0'(x) = -J_1(x)$.

The constant term in (6) has a somewhat different appearance from that

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[2] The customary designation, $j_{\alpha+1,n}$, is avoided in the limits of integration for typographical reasons.

[3] The principal term in (6) is due to T. H. Gronwall [1, 2], whose remainder term was $o(n^{\frac{1}{2}})$. His proofs were rather complicated; simpler ones were devised by G. Szegö [7, 8].

The constant term in (6) was suggested to me by Szegö, along with the outline of a proof. (The value I announced in abstract 544-18, *Notices of the American Mathematical Society*, vol. 5, 1958, p. 212, is incorrect.) A proof of (6) can be given without the full argumentation needed for (2); (15) and (18) can be joined together directly through the familiar asymptotic formula applied to $J_0(x)$ and $J_1(x)$ without the intervention of the Lemma of § 2.

which is obtained on putting $\alpha = 0$ in (4). The transition is made by rewriting the second infinite series in (4) in this case as

$$\lim_{n\to\infty}\{M_1(0) + \cdots + M_n(0) - (2^{\frac{1}{2}}/\pi^{\frac{3}{2}})\int_0^{j_{1n}} x^{-\frac{1}{2}}\,dx\}$$
$$= \lim_{n\to\infty}\{M_1(0) + \cdots + M_n(0) - (2^{\frac{3}{2}}/\pi)n^{\frac{1}{2}}\},$$

in view of (31) below, since, for $\alpha = 0$,

$$\int_0^{j_1} x^{\alpha} J_{\alpha+1}(x)\,dx = \int_0^{j_{11}} J_1(x)\,dx = J_0(0) - J_0(j_{11})$$
$$= 1 + M_1(0).$$

The more precise version of [5, (4)] is:

$$L_n(-\tfrac{1}{2}, \beta) = (4/\pi^2)\log n + C_\beta + O(n^{-1}\log n) + O(n^{-\beta-\frac{3}{2}}), \tag{7}$$

where

$$C_\beta = (4/\pi^2)\log 2 + (2/\pi)\int_0^1 \theta^{-1}\sin\theta\,d\theta$$
$$- (4/\pi^2)\int_0^{\frac{1}{2}\pi}(\{1 - (\cos\theta)^{\beta+\frac{1}{2}}\}/\sin\theta)\,d\theta \tag{8}$$
$$- (2/\pi)\int_1^\infty \theta^{-1}\{(2/\pi) - |\sin\theta|\}d\theta.$$

Putting $\beta = -\frac{1}{2}$ in (7) gives rise to the case of Tchebycheff polynomials (equivalent to ordinary Fourier series; cf. footnote 6 in [5]). In this case (7) and (8) assume the form found for the Lebesgue constants for Fourier series in [3], except for the remainder term.

The discussion of the cases considered here ($\alpha = -\frac{1}{2}$; $-\frac{1}{2} < \alpha < \frac{1}{2}$) is based on (20), a representation of $L_n(\alpha, \beta)$ in terms of Bessel functions. This formula requires for its justification a Lemma permitting the substitution of one (real) solution of Bessel's differential equation for another in certain integrals. This provides the content of § 2. The basic representation (20) is then established in § 3.

Thereafter the cases are handled separately: $\alpha = -\frac{1}{2}$ in § 4 and $-\frac{1}{2} < \alpha < \frac{1}{2}$, $\alpha - \beta < 1$ in § 5. The case $-1 < \alpha < -\frac{1}{2}$ [9, § 20], in which $L_n(\alpha, \beta)$ is bounded, can be handled by the simpler formula (15). This is left to the reader.

2. A lemma on Bessel functions. In deriving (20), the following result is needed. Its final sentence justifies replacing one Bessel function by

another in the integrals involved, without altering either the value of the limit or the magnitudes of the remainder terms.

LEMMA.[4] *For* $0<\epsilon<\pi$, $\nu>-1$, *define*

$$S_N(\alpha,\beta)=N^{\alpha+1}\int_{\pi-\epsilon}^{\pi-1/N}(\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\tfrac{1}{2}\theta)^{\frac{1}{2}}\,|\,\mathcal{C}_\nu(N\theta,\delta)\,|\,d\theta \tag{9}$$

and

$$S_N'(\alpha,\beta)=N^{\alpha+1}\int_{\pi-\epsilon}^{\pi}(\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\tfrac{1}{2}\theta)^{\frac{1}{2}}\,|\,\mathcal{C}_\nu(N\theta,\delta)\,|\,d\theta, \tag{10}$$

where[5] $\mathcal{C}_\nu(x,\delta)=J_\nu(x)\cos\pi\delta-Y_\nu(x)\sin\pi\delta$.

Then

$$S_N(\alpha,\beta)=(2/\pi^{\frac{3}{2}})N^{\alpha+\frac{1}{2}}\int_{\pi-\epsilon}^{\pi-1/N}(\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}\,d\theta+O(N^{\alpha-\frac{1}{2}})+O(N^{\alpha-\beta-1}), \tag{11}$$

and

$$S_N'(\alpha,\beta)=(2/\pi^{\frac{3}{2}})N^{\alpha+\frac{1}{2}}\int_{\pi-\epsilon}^{\pi}(\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}\,d\theta+O(N^{\alpha-\frac{1}{2}})+O(N^{\alpha-\beta-1}). \tag{12}$$

In both (11) *and* (12), *the expressions on the right are independent of* (*fixed*) ν *and* δ.

Proof of (11). Hankel's asymptotic formula [11, p. 488] for $\mathcal{C}_\nu(N\theta,\delta)$ shows that, for $\theta\geqq\pi-\epsilon$, $(\frac{1}{2}\theta)^{\frac{1}{2}}\mathcal{C}_\nu(N\theta,\delta)=(\pi N)^{-\frac{1}{2}}\cos(N\theta+\gamma)+O(N^{-\frac{3}{2}})$, where $\gamma=\delta-\frac{1}{2}\nu\pi-\pi/4$. Using it in (9), we see that its remainder term contributes to $S_N(\alpha,\beta)$ an amount which is $O(N^{\alpha-\frac{1}{2}})$.

Thus, (11) will be established once it is shown that $|\cos(N\theta+\gamma)|$ can be replaced by its mean value $2/\pi$ with an additive error $O(N^{\alpha-\frac{1}{2}})+O(N^{\alpha-\beta-1})$. This, in turn, is an immediate consequence of Theorem 2.1 of [4, p. 90], whose hypotheses can be verified readily for the case at hand. (Cf. [5, § 3, Remark].)

Proof of (12). As in the proof of (11), it is clear that the remainder term in Hankel's asymptotic formula contributes only $O(N^{\alpha-\frac{1}{2}})$ to $S_N'(\alpha,\beta)$. To complete the proof of (12), therefore, it is plainly sufficient to show that

[4] This Lemma is valid without any restriction on α, provided $\beta>-\frac{3}{2}$, *i. e.*, under broader conditions than the requirements $\alpha>-1$, $\beta>-1$ placed throughout both Parts.

[5] This is a minor adaptation of the usual notation for real cylinder functions; $Y_\nu(x)$ is the Bessel function of second kind and ν-th order.

the contribution of the integral over $(\pi - 1/N, \pi)$ involving the principal term does not exceed the error terms stated in (12); this is a straightforward calculation, which is left to the reader.

3. Representation of $L_n(\alpha, \beta)$ in terms of Bessel functions. In this section, the integrand in (1) is replaced by an asymptotic representation (20) in terms of Bessel functions, on which the proofs of (2) and (7) are based. This change can be accomplished by means of an asymptotic formula, due to G. Szegö, of "Hilb type" [10, p. 191, Theorem 8.21.12]. However, this requires decomposing the integral (1) into the sum of two integrals $(0 < \epsilon < \pi)$

$$L_n(\alpha, \beta) = \int_0^{\pi-\epsilon} + \int_{\pi-\epsilon}^{\pi} \equiv L_{n1}(\alpha, \beta) + L_{n2}(\alpha, \beta), \tag{13}$$

where the factor preceding the integral sign in (1) is understood to precede each of the integral signs in (13).

Szegö's formula can be applied directly to the integrand of $L_{n1}(\alpha, \beta)$. Doing so yields

$$\begin{aligned}(\sin \tfrac{1}{2}\theta)^{2\alpha+1}&(\cos \tfrac{1}{2}\theta)^{2\beta+1} P_n^{(\alpha+1,\beta)}(\cos\theta) \\ &= (\Gamma(n+\alpha+2)/N^{\alpha+1}n!)(\sin \tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos \tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\tfrac{1}{2}\theta)^{\frac{1}{2}} J_{\alpha+1}(N\theta) \\ &\quad + \begin{cases} \theta^{\alpha+\frac{1}{2}} O(n^{-\frac{3}{2}}), & 1/n \leqq \theta \leqq \pi - \epsilon, \\ \theta^{2\alpha+3} O(n^{\alpha+1}), & 0 < \theta \leqq 1/n, \end{cases}\end{aligned} \tag{14}$$

where $N = n + \frac{1}{2}(\alpha + \beta + 2)$, a notation which is retained throughout.

The remainder term in (14) contributes $O(n^{\alpha-\frac{1}{2}})$ to $L_n(\alpha, \beta)$, from (1), as may be seen from Stirling's formula.

Next, when the factor not involving θ in the principal term of (14) is combined with the factor preceding the integral sign in (13) [the same as in (1)], the resulting product is

$$\{\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+2)\}/\{N^{\alpha+1}n!\Gamma(\alpha+1)\Gamma(n+\beta+1)\}.$$

This is, from Stirling's formula, $N^{\alpha+1}/\Gamma(\alpha+1) + O(n^{\alpha})$, since $N - n$ is a constant.

Thus,

$$\begin{aligned}L_{n1}(\alpha, \beta) &= \{N^{\alpha+1}/\Gamma(\alpha+1) + O(n^{\alpha})\} \\ &\quad \cdot \int_0^{\pi-\epsilon} (\sin \tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos \tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\tfrac{1}{2}\theta)^{\frac{1}{2}} \,|\, J_{\alpha+1}(N\theta) \,|\, d\theta + O(n^{\alpha-\frac{1}{2}}).\end{aligned} \tag{15}$$

In $L_{n2}(\alpha,\beta)$ it is helpful to replace θ by $\pi-\phi$, so that in view of [10, p. 58, (4.1.3)],

$$L_{n2}(\alpha,\beta)=(\Gamma(n+\alpha+\beta+2)/\Gamma(\alpha+1)\Gamma(n+\beta+1)) \cdot\int_0^{\epsilon}(\sin\tfrac{1}{2}\phi)^{2\beta+1}(\cos\tfrac{1}{2}\phi)^{2\alpha+1}\,|\,P_n^{(\beta,\alpha+1)}(\cos\phi)\,|\,d\phi. \tag{16}$$

In this case Szegö's formula [10, p. 191, Theorem 8.12.12] becomes

$$\begin{aligned}&(\sin\tfrac{1}{2}\phi)^{2\beta+1}(\cos\tfrac{1}{2}\phi)^{2\alpha+1}P_n^{(\beta,\alpha+1)}(\cos\phi)\\ &\quad=(N^{-\beta}\Gamma(n+\beta+1)/n!)\,(\sin\tfrac{1}{2}\phi)^{\beta+\frac{1}{2}}(\cos\tfrac{1}{2}\phi)^{\alpha-\frac{1}{2}}(\tfrac{1}{2}\phi)^{\frac{1}{2}}J_\beta(N\phi)\\ &\qquad+\begin{cases}\phi^{\beta+\frac{3}{2}}O(n^{-\frac{3}{2}}), & 1/n\leqq\phi\leqq\pi-\epsilon,\\ \phi^{2\beta+3}O(n^{\beta}), & 0<\phi\leqq 1/n.\end{cases}\end{aligned} \tag{17}$$

As in the case of $L_{n1}(\alpha,\beta)$, there is no difficulty in showing that the error terms in (17) contribute $O(n^{\alpha-\beta-3})+O(n^{\alpha-\frac{1}{2}})=O(n^{\alpha-\frac{1}{2}})$ to $L_{n2}(\alpha,\beta)$.

Furthermore, combining the ϕ-free factor in the principal term of (17) with the factor preceding the integral sign in $L_{n2}(\alpha,\beta)$ gives rise to a factor which is $N^{\alpha+1}/\Gamma(\alpha+1)+O(n^{\alpha})$, in the same manner as before.

Reverting now to θ, (17) shows thus that

$$L_{n2}(\alpha,\beta)=\{N^{\alpha+1}/\Gamma(\alpha+1)+O(n^{\alpha})\} \cdot\int_{\pi-\epsilon}^{\pi}(\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\tfrac{1}{2}\{\pi-\theta\})^{\frac{1}{2}}\,|\,J_\beta(N\{\pi-\theta\})\,|\,d\theta+O(n^{\alpha-\frac{1}{2}}). \tag{18}$$

We wish now to replace $(\frac{1}{2}\{\pi-\theta\})^{\frac{1}{2}}\,|\,J_\beta(N\{\pi-\theta\})\,|$ by $(\frac{1}{2}\theta)^{\frac{1}{2}}\,|\,J_{\alpha+1}(N\theta)\,|$ in (18). This can be justified by means of (a) Hankel's asymptotic formula [11, p. 488] and (b) the Lemma of § 2.

It is convenient to consider separately the cases $\beta=-\frac{1}{2}$ and $\beta\neq-\frac{1}{2}$.

For $\beta=-\frac{1}{2}$, [11, pp. 55, 64],

$$\begin{aligned}(\pi-\theta)^{\frac{1}{2}}\,|\,J_{-\frac{1}{2}}(N\{\pi-\theta\})\,|&=(2/\pi)^{\frac{1}{2}}N^{-\frac{1}{2}}\,|\cos N\theta\,|\\ &=\theta^{\frac{1}{2}}\,|\,J_{-\frac{1}{2}}(N\theta)\,|=\theta^{\frac{1}{2}}\,|\,Y_{\frac{1}{2}}(N\theta)\,|=\theta^{\frac{1}{2}}\,|\,\mathscr{C}_{\frac{1}{2}}(N\theta,-\tfrac{1}{2})\,|\end{aligned}$$

so that

$$(\tfrac{1}{2}\{\pi-\theta\})^{\frac{1}{2}}\,|\,J_{-\frac{1}{2}}(N\{\pi-\theta\})\,|=(\tfrac{1}{2}\theta)^{\frac{1}{2}}\,|\,\mathscr{C}_{\frac{1}{2}}(N\theta,-\tfrac{1}{2})\,|. \tag{19}$$

Using (19) and the Lemma of § 2 (whose final sentence permits replacing $\mathscr{C}_{\frac{1}{2}}(N\theta,-\frac{1}{2})$ by $J_{\alpha+1}(N\theta)$), the integrand on the right of (18) becomes identical with that in (15) with an error of $O(n^{\alpha-\beta-1})+O(n^{\alpha-\frac{1}{2}})$.

Hence, for $\beta=-\frac{1}{2}$, we have from (13),

$$L_n(\alpha,\beta)=\{N^{\alpha+1}/\Gamma(\alpha+1)+O(n^{\alpha})\} \cdot\int_0^{\pi}(\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\tfrac{1}{2}\theta)^{\frac{1}{2}}\,|\,J_{\alpha+1}(N\theta)\,|\,d\theta+O(n^{\alpha-\beta-1})+O(n^{\alpha-\frac{1}{2}}). \tag{20}$$

The same formula is valid also when $\beta \neq -\frac{1}{2}$. To show this, it is helpful to decompose $L_{n2}(\alpha, \beta)$:

$$L_{n2}(\alpha,\beta) = \{N^{\alpha+1}/\Gamma(\alpha+1) + O(n^{\alpha})\}\left\{\int_{\pi-\epsilon}^{\pi-1/N} + \int_{\pi-1/N}^{\pi}\right\} \tag{21}$$
$$\equiv L_{n2}'(\alpha,\beta) + L_{n2}''(\alpha,\beta).$$

Now,

$$L_{n2}''(\alpha,\beta) = O(n^{\alpha+1})\int_{\pi-1/N}^{\pi} (\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\pi-\theta)^{\frac{1}{2}} \mid J_{\beta}(N\{\pi-\theta\})\mid d\theta$$
$$= O(n^{\alpha+1})\int_{0}^{1/N} (\sin\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}\theta^{\frac{1}{2}} \mid J_{\beta}(N\theta)\mid d\theta$$
$$= O(n^{\alpha+\beta+1})\int_{0}^{1/N} \theta^{2\beta+1}\, d\theta = O(n^{\alpha-\beta-1}),$$

since $J_{\beta}(N\theta) \sim N^{\beta}\theta^{\beta}$ for $N\theta$ small [10, p. 16, (1.71.10)].

To L_{n2}', Hankel's asymptotic formula [11, p. 488] may be applied. This implies, for $\pi-\epsilon \leqq \theta \leqq \pi - 1/N$,

$$(\tfrac{1}{2}\{\pi-\theta\})^{\frac{1}{2}} \mid J_{\beta}(N\{\pi-\theta\})\mid = (\tfrac{1}{2}\theta)^{\frac{1}{2}} \mid \mathcal{C}_{\beta+1}(N\theta,\beta)\mid + (\pi-\theta)^{-1}O(N^{-\frac{3}{2}}), \tag{22}$$

a result which takes the more precise form (19) when $\beta = -\frac{1}{2}$.

For $\beta \neq -\frac{1}{2}$, the remainder term in (22) contributes to $L_{n2}'(\alpha,\beta)$ an amount which is $O(n^{\alpha-\beta-1}) + O(n^{\alpha-\frac{1}{2}})$.

Thus,

$$L_{n2}(\alpha,\beta) = \{N^{\alpha+1}/\Gamma(\alpha+1) + O(n^{\alpha})\}$$
$$\cdot \int_{\pi-\epsilon}^{\pi-1/N} (\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\tfrac{1}{2}\theta)^{\frac{1}{2}} \mid \mathcal{C}_{\beta+1}(N\theta,\beta)\mid d\theta$$
$$+ O(n^{\alpha-\beta-1}) + O(n^{\alpha-\frac{1}{2}}).$$

The expression corresponding to the right member of this last equation with the limits of integration replaced by $\pi - 1/N$ and π is $O(n^{\alpha-\beta-1}) + O(n^{\alpha-\frac{1}{2}})$, as shown by the same argument used to establish (12). Hence,

$$L_{n2}(\alpha,\beta) = \{N^{\alpha+1}/\Gamma(\alpha+1) + O(n^{\alpha})\}$$
$$\cdot \int_{\pi-\epsilon}^{\pi} (\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\tfrac{1}{2}\theta)^{\frac{1}{2}} \mid \mathcal{C}_{\beta+1}(N\theta,\beta)\mid d\theta$$
$$+ O(n^{\alpha-\beta-1}) + O(n^{\alpha-\frac{1}{2}}).$$

To this the Lemma of § 2 applies, showing that the replacement of $\mid \mathcal{C}_{\beta+1}(N\theta,\beta)\mid$ by $\mid J_{\alpha+1}(\theta)\mid$ in the integrand does not affect the value of $L_{n2}(\alpha,\beta)$ beyond the error stated in (12). This substitution made, (13) shows that (20) *is valid also for* $\beta \neq -\frac{1}{2}$, *as well as for* $\beta = -\frac{1}{2}$.

4. The case $\alpha = -\frac{1}{2}$; proof of (7). When $\alpha = -\frac{1}{2}$, (20) becomes [11, p. 55]

$$L_n(-\tfrac{1}{2},\beta) = \{2/\pi + O(n^{-1})\} \cdot \int_0^{\frac{1}{2}\pi} \{(\cos\theta)^{\beta+\frac{1}{2}} \,|\sin 2N\theta\,|/\sin\theta\}d\theta + O(n^{-\beta-\frac{3}{2}}) + O(n^{-1}). \tag{23}$$

The desired asymptotic expression for $L_n(-\frac{1}{2},\beta)$ is obtained by comparing the integral on the right of (23) with the extension to non-integral N of the Lebesgue constants for Fourier series (cf. [4, Theorem 4.1, p. 96]), essentially $L_n(-\frac{1}{2},-\frac{1}{2})$. Obviously, it suffices to consider $\beta \neq -\frac{1}{2}$ below.

In order to include the case in which $\beta+\frac{1}{2}$ is negative, it is convenient to decompose the integral in question into the sum of two integrals, over $(0, \frac{1}{2}\pi - \pi/(2N))$ and $(\frac{1}{2}\pi - \pi/(2N), \frac{1}{2}\pi)$, say $\rho_{n1}(-\frac{1}{2},\beta)$ and $\rho_{n2}(-\frac{1}{2},\beta)$, respectively.

It is easy to show that $\rho_{n2}(-\frac{1}{2},\beta) = O(N^{-\beta-\frac{3}{2}})$, an amount which can be absorbed in the error terms already found in (23).

Hence it is sufficient to consider $\rho_{n1}(-\frac{1}{2},\beta)$.

Now,

$$\begin{aligned}(2/\pi)\int_0^{\frac{1}{2}\pi-\pi/(2N)} &(\{1-(\cos\theta)^{\beta+\frac{1}{2}}\}|\sin 2N\theta\,|/\sin\theta)\,d\theta \\ &= (4/\pi^2)\int_0^{\frac{1}{2}\pi-\pi/(2N)} (\{1-(\cos\theta)^{\beta+\frac{1}{2}}\}/\sin\theta)\,d\theta + O(N^{-\beta-\frac{3}{2}}) + O(N^{-1})\end{aligned} \tag{24}$$

from Theorem 2.1 of [4], since the mean value of $|\sin\theta|$ is $2/\pi$.

Thus,

$$\begin{aligned}L_n(-\tfrac{1}{2},\beta) &= (2/\pi)\int_0^{\frac{1}{2}\pi-\pi/(2N)} (|\sin 2N\theta\,|/\sin\theta)\,d\theta \\ &\quad -(4/\pi^2)\int_0^{\frac{1}{2}\pi-\pi/(2N)} (\{1-(\cos\theta)^{\beta+\frac{1}{2}}\}/\sin\theta)d\theta + O(N^{-\beta-\frac{3}{2}}) + O(N^{-1}),\end{aligned} \tag{25}$$

and from this it follows easily that

$$\begin{aligned}L(-\tfrac{1}{2},\beta) &= \{(2/\pi) + O(n^{-1})\}\int_0^{\frac{1}{2}\pi} (|\sin 2N\theta\,|/\sin\theta)d\theta \\ &\quad -(4/\pi^2)\int_0^{\frac{1}{2}\pi} (\{1-(\cos\theta)^{\beta+\frac{1}{2}}\}/\sin\theta)d\theta + O(N^{-\beta-\frac{3}{2}}) + O(N^{-1}).\end{aligned} \tag{26}$$

The first integral on the right in (26) is the extension to non-integral values of the n-th Lebesgue constant in Fourier series. This is directly covered by Theorem 4.1 of [4, p. 96], whose use completes the proof of (7) since $N-n$ is a constant, namely $\frac{1}{2}(\beta+\frac{3}{2})$.

5. The case $-\frac{1}{2} < \alpha < \frac{1}{2}$, $\alpha - \beta < 1$; proof of (2). In this case, the remainder term in the factor of the integral on the right of (20) induces a contribution to $L_n(\alpha, \beta)$ which is $O(n^{\alpha-\frac{1}{2}})$ and hence absorbable in the error terms already present in (20). This can be shown by estimates of the same type as have already been made frequently in previous sections. In carrying them out, it is convenient to write $L_n(\alpha, \beta) = L_{n1}(\alpha, \beta) + L_{n2}(\alpha, \beta)$ again and to use the representation (18) for $L_{n2}(\alpha, \beta)$. Alternatively, this can be held in abeyance, to appear incidentally in the forthcoming estimate of the integral on the right of (20).

In any event, the factor of that integral may as well be taken as $N^{\alpha+1}/\Gamma(\alpha+1)$ from the outset, thus simplifying the notation. This done, (20) can be written as follows:

$$L_n(\alpha, \beta) = \{N^{\alpha+1}/\Gamma(\alpha+1)\} \int_0^{\pi} f_{\alpha\beta}(\theta)\theta^{-\alpha} \,|\, J_{\alpha+1}(N\theta) \,|\, d\theta + O(n^{\alpha-\frac{1}{2}}) + O(n^{\alpha-\beta-1}), \tag{27}$$

where

$$f_{\alpha\beta}(\theta) = (\tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta)^{\frac{1}{2}}(\theta \sin \tfrac{1}{2}\theta)^{\alpha}(\cos \tfrac{1}{2}\theta)^{\beta}. \tag{28}$$

Now [11, p. 45], $D_x\{x^{-\alpha}J_\alpha(x)\} = -x^{-\alpha}J_{\alpha+1}(x)$. Hence, integrating by parts and writing $x = N\theta$, $N\theta_k = j_{\alpha+1,k}$ with $0 \leqq \theta < \pi$, $0 \leqq \theta_k < \pi$, we have

$$N \int_{\theta_k}^{\theta_{k+1}} f_{\alpha\beta}(\theta)\theta^{-\alpha}J_{\alpha+1}(N\theta)\,d\theta = f_{\alpha\beta}(\theta_k)\theta_k{}^{-\alpha}J_\alpha(j_{\alpha+1,k}) - f_{\alpha\beta}(\theta_{k+1})\theta_{k+1}{}^{-\alpha}J_\alpha(j_{\alpha+1,k+1}) + \int_{\theta_k}^{\theta_{k+1}} f_{\alpha\beta}'(\theta)\theta^{-\alpha}J_\alpha(N\theta)\,d\theta,$$

where $J_{\alpha+1}(N\theta)$ is of constant sign $(-1)^k$ in the interval of integration.

Furthermore,

$$\begin{aligned} N^{\alpha+1} &\int_0^{\theta_1} f_{\alpha\beta}(\theta)\theta^{-\alpha} \,|\, J_{\alpha+1}(N\theta) \,|\, d\theta \\ &= 2^{-\alpha} \int_0^{j_1} \{(t/[2N])/\tan(t/[2N])\}^{\frac{1}{2}}\{\sin(t/[2N])/(t/[2N])\}^{\alpha} \\ &\quad \cdot \{\cos(t/[2N])\}^{\beta} t^{\alpha} J_{\alpha+1}(t)\,dt \\ &= 2^{-\alpha} \int_0^{j_1} t^{\alpha}J_{\alpha+1}(t)\,dt + O(n^{-2}), \end{aligned}$$

since $J_{\alpha+1}(t) > 0$ in the interval of integration, and the factor of $t^{\alpha}J_{\alpha+1}(t)$ in the next to last integral is an even analytic function assuming the value 1 at the origin and hence representable as $1 + O(t^2/N^2)$.

Thus, (27) can be written as

$$
\begin{aligned}
L_n(\alpha,\beta) = (2^{-\alpha}/\Gamma(\alpha+1))\{ & \int_0^{j_1} t^\alpha J_{\alpha+1}(t)\,dt \\
& + 2\sum_{k=2}^{[N]-1} g_{\alpha\beta}(\theta_k)M_k(\alpha) + g_{\alpha\beta}(\theta_1)M_1(\alpha) \\
& + g_{\alpha\beta}(\theta_{[N]})M_{[N]}(\alpha)\} + (N^\alpha/\Gamma(\alpha+1))\int_{\theta_{[N]}}^{\pi} f_{\alpha\beta}(\theta)\theta^{-\alpha}\,|\,J_{\alpha+1}(N\theta)|\,d\theta \\
& + (2^{-\alpha}N^\alpha/\Gamma(\alpha+1))\sum_{k=1}^{[N]-1}(-1)^k\int_{\theta_k}^{\theta_{k+1}} f_{\alpha\beta}'(\theta)\theta^{-\alpha}J_\alpha(N\theta)\,d\theta \\
& + O(n^{\alpha-\frac{1}{2}}) + O(n^{\alpha-\beta-1}),
\end{aligned}
\tag{29}
$$

where $[N]$ denotes, as usual, the greatest integer $\leqq N$, and

$$
g_{\alpha\beta}(\theta) = 2^\alpha f_{\alpha\beta}(\theta)\theta^{-2\alpha}. \tag{30}
$$

Now we need the asymptotic estimates

$$
\begin{cases} j_{\alpha+1,k} = (k+\frac{1}{2}\alpha+\frac{1}{4})\pi + O(k^{-1})\,; \\ M_k(\alpha) = (2/\pi)^{\frac{1}{2}} j_{\alpha+1,k}{}^{\alpha-\frac{1}{2}} + O(j_{\alpha+1,k}{}^{\alpha-\frac{3}{2}}), \end{cases} \tag{31}
$$

the first of which follows from [11, p. 506], the second becoming then a consequence of [11, p. 488] or [10, p. 15, (1.71.7)].

Now, $J_{\alpha+1}(t) = O(t^{-\frac{1}{2}})$ and $\theta_{[N]} = \pi - O(n^{-1})$, so that

$$
N^\alpha \int_{\theta_{[N]}}^{\pi} f_{\alpha\beta}(\theta)\theta^{-\alpha}\,|\,J_{\alpha+1}(N\theta)|\,d\theta = O(n^{\alpha-1}), \tag{32}
$$

since the left member is

$$
\begin{aligned}
O(n^{\alpha-\frac{1}{2}}) \int_{\theta_{[N]}}^{\pi} (\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}\,d\theta &= O(n^{\alpha-\frac{1}{2}}) \int_0^{O(1/n)} \theta^{\beta+\frac{1}{2}}\,d\theta \\
&= O(n^{\alpha-\beta-2}) = O(n^{\alpha-1}),
\end{aligned}
$$

since $\beta > -1$.

Furthermore,

$$
g_{\alpha\beta}(\theta_1)M_1(\alpha) = M_1(\alpha) + O(n^{-2}), \tag{33}
$$

since $g_{\alpha\beta}(\theta)$ is an even analytic function assuming the value 1 at the origin and $\theta_1 = j_1/N$, while

$$
g_{\alpha\beta}(\theta_{[N]})M_{[N]}(\alpha) = O(n^{\alpha-\beta-1}), \tag{34}
$$

a magnitude already included in the error in (29).

The estimate (34) follows from (30) and (31), since $\theta_{[N]} = \pi - O(n^{-1})$, while

$$g_{\alpha\beta}(\theta_{[N]})M_{[N]}(\alpha) = O(\{\cot \tfrac{1}{2}(\pi - n^{-1})\}^{\frac{1}{2}}\{\sin(1/n)\}^{\beta})O(n^{\alpha-\frac{1}{2}})$$
$$= O(n^{-\frac{1}{2}-\beta+\alpha-\frac{1}{2}}) = O(n^{\alpha-\beta-1}).$$

Next we consider

$$D_n \equiv (N^{\alpha}/\Gamma(\alpha+1))\sum_{k=1}^{[N]-1}(-1)^k \int_{\theta_k}^{\theta_{k+1}} f_{\alpha\beta}'(\theta)\theta^{-\alpha}J_{\alpha}(N\theta)\,d\theta, \tag{35}$$

in the form

$$D_n = (N^{2\alpha-1}/\Gamma(\alpha+1))\sum_{k=1}^{[N]-1}(-1)^k \int_{j_k}^{j_{k+1}} f_{\alpha\beta}'(x/N)x^{-\alpha}J_{\alpha}(x)\,dx.$$

From (28) it is clear that $f_{\alpha\beta}(\theta)$ is $2^{-\alpha}\theta^{2\alpha}$ times an even analytic function, regular in a neighborhood of the origin, whose value at the origin is 1. Thus,

$$f_{\alpha\beta}(\theta) = 2^{-\alpha}\theta^{2\alpha}\{1 + O(\theta^2)\}, \qquad (\theta \to 0),$$

whence

$$f_{\alpha\beta}'(\theta) = 2^{-\alpha}(2\alpha)\theta^{2\alpha-1} + O(\theta^{2\alpha+1}), \qquad (\theta \to 0).$$

Hence

$$D_n = (2^{-\alpha}/\Gamma(\alpha+1))\sum_{k=1}^{[N]-1}(-1)^k \int_{j_k}^{j_{k+1}} \{2\alpha + O(x^2/N^2)\}x^{\alpha-1}J_{\alpha}(x)\,dx.$$

The remainder term in this integrand contributes to D_n an amount which is

$$O(N^{-2})\sum_{k=1}^{[N]-1}\int_{j_k}^{j_{k+1}} x^{\alpha+1}\,|J_{\alpha}(x)|\,dx$$
$$= O(N^{-2})\int_{j_1}^{j_{[N]}} x^{\alpha+1}\,|J_{\alpha}(x)|\,dx$$
$$= O(N^{-2})\int_{j_1}^{j_{[N]}} x^{\alpha+\frac{1}{2}}\,dx = O(N^{-2})j_{\alpha+1,[N]}{}^{\alpha+\frac{3}{2}}$$
$$= O(N^{\alpha-\frac{1}{2}}) = O(n^{\alpha-\frac{1}{2}}),$$

since $J_{\alpha}(x) = O(x^{-\frac{1}{2}})$, and $j_{\alpha+1,[N]} = O(N)$ according to (31).

So,

$$D_n = (2^{1-\alpha}/\Gamma(\alpha+1))\sum_{k=1}^{[N]-1}(-1)^k \int_{j_k}^{j_{k+1}} x^{\alpha-1}J_{\alpha}(x)\,dx + O(n^{\alpha-\frac{1}{2}}).$$

Now,

$$\left|\int_{j_k}^{j_{k+1}} x^{\alpha-1}J_\alpha(x)\,dx\right| = O(1)\int_{j_k}^{j_{k+1}} x^{\alpha-\frac{3}{2}}\,dx$$
$$= O(1)\{j_{\alpha+1,k+1}{}^{\alpha-\frac{1}{2}} - j_{\alpha+1,k}{}^{\alpha-\frac{1}{2}}\} = O(1)\{(k+1)^{\alpha-\frac{1}{2}} - k^{\alpha-\frac{1}{2}}\}$$
$$= O(k^{\alpha-\frac{3}{2}}),$$

where we have used again that $J_\alpha(x) = O(x^{-\frac{1}{2}})$ and (31), and, for the final estimate, the mean-value theorem of the differential calculus.

Thus,

$$D_n = (2^{1-\alpha}\alpha/\Gamma(\alpha+1))\sum_{k=1}^{\infty}(-1)^k\int_{j_k}^{j_{k+1}} x^{\alpha-1}J_\alpha(x)\,dx + O(n^{\alpha-\frac{1}{2}}), \tag{36}$$

where the infinite series is absolutely convergent, since each term is $O(k^{\alpha-\frac{3}{2}})$ and $\alpha < \frac{1}{2}$.

There remains only to consider the first (finite) sum on the right in (29). For this purpose we write

$$2\sum_{k=2}^{[N]-1} g_{\alpha\beta}(\theta_k)M_k(\alpha)$$
$$= 2\sum_{k=2}^{[N]-1} g_{\alpha\beta}(\theta_k)\{M_k(\alpha) - (2^{\frac{1}{2}}/\pi^{\frac{3}{2}})\int_{j_{k-1}}^{j_k} x^{\alpha-\frac{1}{2}}\,dx\} \tag{37}$$
$$+ (2/\pi)^{\frac{3}{2}}\sum_{k=2}^{[N]-1}\int_{j_{k-1}}^{j_k} g_{\alpha\beta}(\theta_k)x^{\alpha-\frac{1}{2}}\,dx.$$

Here we note that $g_{\alpha\beta}(\theta)$ is an even analytic function, regular in a neighborhood of the origin, which assumes the value 1 at the origin. Thus, from (31), $g_{\alpha\beta}(\theta_k) = 1 + O(k^2/n^2)$, $k = 2, \cdots, [N]-1$.

We consider first the contribution induced by the remainder $O(k^2/n^2)$ to the first sum in the right member of (37). In that sum, the expression in braces is $O(k^{\alpha-\frac{3}{2}})$, from (31), so that the contribution now under discussion is

$$O(n^{-2})\sum_{k=2}^{[N]-1} k^2k^{\alpha-\frac{3}{2}} = O(n^{-2})\sum_{k=2}^{[N]-1} k^{\alpha+\frac{1}{2}} = O(n^{\alpha-\frac{1}{2}}).$$

Thus, $g_{\alpha\beta}(\theta_k)$ can be replaced in the first sum on the right in (37) by 1, with error $O(n^{\alpha-\frac{1}{2}})$, an amount which can be absorbed in the error terms already present in (29).

Passing to the final sum on the right of (37), we note that $g_{\alpha\beta}(\theta_k) = g_{\alpha\beta}(\theta) + O(n^{-1})$, as can be inferred from the mean-value theorem of differential calculus, since $g_{\alpha\beta}(\theta) = 1 + O(n^{-2})$ and $g_{\alpha\beta}{}'(\theta) = O(n^{-1})$. Re-

placing $g_{\alpha\beta}(\theta_k)$ by $g_{\alpha\beta}(\theta)+O(n^{-1})$, we find that the error term induces a contribution of

$$O(n^{-1})\sum_{k=2}^{[N]-1}\int_{j_{k-1}}^{j_k}x^{\alpha-\frac{1}{2}}\,dx=O(n^{-1})\int_{j_1}^{j_{[N]-1}}x^{\alpha-\frac{1}{2}}\,dx=O(n^{\alpha-\frac{1}{2}}),$$

since $j_{[N]}=O(N)=O(n)$ from (31).

Hence,

$$\begin{aligned}2\sum_{k=2}^{[N]-1}g_{\alpha\beta}(\theta_k)M_k(\alpha)&=2\sum_{k=2}^{\infty}\{M_k(\alpha)-(2^{\frac{1}{2}}/\pi^{\frac{3}{2}})\int_{j_{k-1}}^{j_k}x^{\alpha-\frac{1}{2}}\,dx\}\\&\quad+(2/\pi)^{\frac{3}{2}}\int_{j_1}^{j_{[N]-1}}g_{\alpha\beta}(\theta)x^{\alpha-\frac{1}{2}}\,dx+O(n^{\alpha-\frac{1}{2}}),\end{aligned}\tag{38}$$

where the passage from the finite sum which arises directly from our discussion of the first finite sum on the right of (37) to the (absolutely convergent) infinite series in (38) leads to an error of only $O(n^{\alpha-\frac{1}{2}})$ since each term of the series is $O(k^{\alpha-\frac{3}{2}})$ from (31).

Now, using (28) and (30) and putting $x=N\theta$,

$$\begin{aligned}&\int_{j_1}^{j_{[N]-1}}g_{\alpha\beta}(\theta)x^{\alpha-\frac{1}{2}}\,dx\\&\quad=2^{\alpha-\frac{1}{2}}N^{\alpha+\frac{1}{2}}\int_{\theta_1}^{\theta_{[N]-1}}(\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}\,d\theta\\&\quad=2^{\alpha-\frac{1}{2}}N^{\alpha+\frac{1}{2}}\int_{0}^{\pi}(\sin\tfrac{1}{2}\theta)^{\alpha-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}\,d\theta+O(n^{\alpha-\frac{1}{2}})\\&\quad=2^{\alpha-\frac{1}{2}}N^{\alpha+\frac{1}{2}}\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\beta+\tfrac{3}{4})/\Gamma(\tfrac{1}{2}\{\alpha+\beta\}+1)+O(n^{\alpha-\frac{1}{2}}).\end{aligned}$$

The replacement of θ_1 and $\theta_{[N]-1}$ by 0 and π, respectively, introduces the stated error $O(n^{\alpha-\frac{1}{2}})$, since $\theta_1=O(n^{-1})$ and $\pi-\theta_{[N]-1}=O(n^{-1})$. The resulting integral is a standard form in the theory of the gamma and beta functions.

Now we substitute in (29) the above, together with (38), as well as (32), (33), (34), (35) and (36). This proves (2) with n replaced by N. Since $N-n$ is a constant, namely $\frac{1}{2}(\alpha+\beta+2)$, the error committed in replacing N by n in (2) is $O(n^{\alpha-\frac{1}{2}})$, an amount already present there.

This completes the proof of (2).

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SELF-LINKED SUBGROUPS OF SEMIGROUPS.*

By M. L. CURTIS.[1]

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1. Introduction. A. D. Wallace posed the following question. If euclidean 3-space E^3 has a continuous associative multiplication with a unit, and C is a circle subgroup, does it follow that C is unknotted? The principal result of this note is that C is unknotted if it is tame. The hypothesis that there be a unit is not needed.

A set Y in a space S is said to be self-linked if Y is contractible to a point in S, but there does not exist a homotopy $\psi: Y \times I \to S$ such that:

(1) ψ^* is the identity,

(2) $\psi_t(Y) \subset S - Y$ for all $t > 0$,

(3) $\psi_1(Y)$ is a point.

We prove a basic theorem about compact self-linked subgroups of locally compact semigroups and use this result to prove that, in the situation of the first paragraph above, C cannot be self-linked.

Next we use a theorem proved by Papakyriakopoulos (and communicated to us by R. H. Fox) to the effect that if a tame simple closed curve in E^3 is not self-linked, then it is unknotted. (Some wild simple closed curves bound disks and hence are clearly not self-linked). The theorem is a slight generalization of Dehn's lemma.

2. Three trivial lemmas.

LEMMA 1. *Let S be a semigroup, e be an idempotent in S, and $K = eSe$. If $x \in K$ has an inverse relative to e, then it has an inverse relative to e in K.*

Proof. Let $xy = e$ with $x \in K$. Then $x = eze$ so that $exe = eze$ and hence $x = exe$. Now $exey = e$ implies that $exeeye = e$, so that $eye \in K$ is also a right inverse for x relative to e.

We note for future reference that $K = eSe$ is precisely the set of all elements of S for which e acts as an identity.

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* Received ——————.

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LEMMA 2. *Let S be a semigroup and e an idempotent in S. Let C be a subgroup of $K = eSe$ such that e is the identity of C. If $x \in K$ and $xC \cap C \neq \phi$ (or $Cx \cap C \neq \phi$), then $x \in C$.*

Proof. Suppose $xr = c$ with r and c in C. Then $xrr^{-1} = cr^{-1}$ so that $xe = cr^{-1}$. Since e is an identity for x, we have $x = cr^{-1} \in C$.

LEMMA 3. *Let S be a topological semigroup with e, C, K as in Lemma 2. If $\alpha: I \to S$ is a path such that $\alpha(0) = e$ and $\alpha(t) \in K - C$ for each $t > 0$, then the homotopy $\phi: C \times I \to S$ defined by $\phi(c,t) = c\alpha(t)$ is such that ϕ_0 is the identity and $\phi(c,t) \in K - C$ for all $t > 0$.*

Proof. The only thing to prove is that $\phi(c,t) \in K - C$ for each $t > 0$. Suppose $c\alpha(t) = d$ with $c, d \in C$. Then $e\alpha(t) = c^{-1}d$ and $\alpha(t) \in K$, so Lemma 2 implies $\alpha(t) \in C$ contradicting an hypothesis.

3. Basic theorem.

THEOREM 1. *Let S be a locally compact topological semigroup with a compact self-linked subgroup C having identity e. Suppose that each point x of $K - C$ ($K = eSe$) can be joined to e by a path α such that $\alpha(0) = e$ and $\alpha(t) \in K - C$ for all $t > 0$. Then K is a topological group.*

Proof. It follows from elementary group theory together with Lemma 1 that K is a group if each $x \in K$ has a right inverse relative to e (which is clearly an idempotent).

Since C is self linked there is a contraction $\pi: C \times I \to S$ of C to a point. Let $H = \pi(C \times I)$. We will show that for each $x \in K$, $xH \cap C \neq \phi$.

Choose a path α from e to x as in the hypotheses of the theorem.. Define a homotopy of C by

$$\theta(c,t) = \begin{cases} \alpha(2t)c & 0 \leqq t \leqq \frac{1}{2} \\ x\pi(c, 2t-1) & \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

We see that $\theta(C,1)$ is a point. Also θ is well defined at $\frac{1}{2}$ because $\alpha(1)c = xc$ and $x\pi(c,0) = xc$. Since C is self-linked there must exist (c_0, t_0) such that $\theta(c_0, t_0) \in C$. By Lemma 3 we must have $t_0 > \frac{1}{2}$. Hence $x\pi(c_0, 2t_0 - 1) \in C$ and $xH \cap C \neq \phi$ as asserted.

Then for $x \in K$ we can choose $y \in H$ such that $xy \in C$. Then xy has an inverse m in C and $xym = e$. By Lemma 1 we have that $eyme$ is an element of K which acts as an inverse for x. Hence K is a group. To show that K is a topological group it remains to show that the inverse map is continuous.

Let $\rho: K \times H \to S$ be the multiplication of S cut down to $K \times H$. Let

$P=\rho^{-1}(C)$. Define $\beta: P\to S$ by $\beta(x,y)=y(xy)^{-1}$ and define $\phi: S\to S$ by $\phi(y)=eye$. For $x\in K$ let $L_x=[(x,y)\,|\,(x,y)\in P]$. Then the inverse map can be described by

$$x\to L_x \xrightarrow{\beta} S \xrightarrow{\phi} K.$$

Hence for each x, $\phi\beta(L_x)$ is a point and clearly β and ϕ are continuous.

We note that since S is locally compact and ϕ is a retraction, K is locally compact. Since C is compact, H is compact. From this it follows that the set function mapping x to L_x is upper semicontinuous. Let x_0^{-1} be the inverse of x_0; i.e. $x_0^{-1}=\phi\beta(L_{x_0})$. Let U be a neighborhood of x_0^{-1}. Then there exists a neighborhood W of L_{x_0} in P such that $\phi\beta(W)\subset U$. Since the $x\to L_x$ mapping is upper semicontinuous, there exists a neighborhood V of x_0 such that for $x\in V$ we have $L_x\cap W\neq\phi$. It follows that $\phi\beta(L_x)$ lies in U for each x, establishing the continuity of the inverse map.

4. Application to E^3.

THEOREM 2. *If E^3 has a continuous associative multiplication and contains a circle subgroup C, then C cannot be self-linked.*

Proof. Assume that C is self-linked. Let e be the identity of C, and let $K=eE^3e$. We will show that K is a topological group, and by Theorem 1 this will follow if we can find paths α form e to points x in $K-C$ with only the end point of α in C.

Now K is arcwise connected since $\phi: E^3\to K$ is a retraction. Given $x\in K-C$, let β be a path from some point c_0 of C to x such that for $t>0$ we have $\beta(t)\in K-C$. Let θ be a path from c_0^{-1} to e in C. Define $\alpha(t)=\beta(t)\theta(t)$, and note that $\alpha(0)=\beta(0)\theta(0)=c_0c_0^{-1}=e$ and that $\alpha(1)=\beta(1)\theta(1)=xe=x$. Also if $t>0$, then $\alpha(t)=\beta(t)\theta(t)$ and $\theta(t)\in C$. If $\alpha(t)\in C$, then we would have a contradiction of our choice of β. Hence K is a topological group.

Now K is a closed subset of E^3. For if y is a limit point of K, then eye is a limit point of precisely the same subsets of K of which y is a limit point, since for each x in K we have that $x=exe$.

Next we show that K is 2-dimensional. Suppose that $\dim K=3$. Then K contains an open set in E^3, and since it is homogeneous, K must be open. It follows that $K=E^3$. Then K is a Lie group with the compact subgroup C, and we have a fibering of E^3 with a compact fiber. This is not possible (see [1] and [3]) so that $\dim K<3$. Now K is contractible since it is a

retract of a contractible space. Thus K contains a singular disk with C as its boundary, showing that $\dim K > 1$, so that $\dim K = 2$.

The existence of the retraction ϕ implies that K is locally contractible. It is clearly locally compact and finite dimensional, so it follows from the theorem on page 185 of [2] that it is a Lie group. As a contractible 2-dimensional Lie group K is a plane. Since $C \subset K$, C cannot be self-linked, and this contradiction establishes the theorem.

COROLLARY. *With the hypotheses of Theorem 2 if C is also tame, then C is unknotted.*

Proof. This follows immediately from Theorem 2 and the theorem of Papakyriakopoulos mentioned in the introduction.

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ON INVOLUTIONS OF THE 3-SPHERE.*

By Morris W. Hirsch and Stephen Smale.[1]

1. Introduction. If T is an orientation reversing involution of the 3-sphere S^3, it follows from the Smith theory [15] that the fixed point set F of T is either a 2-sphere or two points. If F is a 2-sphere, it may be wildly embedded in S^3 [2], in which case T will not be equivalent to a linear involution. (If T_1 and T_2 are involutions on spaces X_1 and X_2 respectively, an *equivalence* between T_1 and T_2 is a homeomorphism $h: X_1 \to X_2$ such that $hT_1 = T_2h$.) On the other hand if $F = S^2$ is tamely embedded, it is easy to show that T is equivalent to a reflection through an equatorial 2-sphere of S^3. The main purpose of this paper is to study the case where F is two points. We will prove

Theorem 1.1. *If $T: S^3 \to S^3$ is an involution with fixed point set F consisting of two points, then T is equivalent to the linear involution: $L: S^3 \to S^3$, $L(x_1, x_2, x_3, x_4) = (x_1, -x_2, -x_3, -x_4)$.*

Here we suppose that S^3 is the unit sphere in Euclidean 4-space with coordinates (x_1, x_2, x_3, x_4). If T is differentiable, then one can obtain actually a differentiable equivalence by our methods.

Theorem 1.1 answers a question of Floyd [6, p. 92].

The problems concerning an orientation-preserving involution on S^3 remain largely unsolved.[2]

As a by-product of the proof of 1.1, the following two theorems are proved:

Theorem 1.2. *Let M be a non-orientable triangulated 3-manifold with an element β of $\pi_1(M)$ such that $\beta^2 = 1$ and β reverses orientation. Then the projective plane P can be embedded in M piecewise linearly.*

Theorem 1.3. *Let T be an involution on a topological 3-manifold M*

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[2] For the 2-sphere these problems have been solved by Eilenberg and Kerekjarto; see S. Eilenberg, *Fund. Math.* XXII (1934), pp. 28-41.

with an isolated fixed point y_0 not in the boundary of M. Then there is an invariant Euclidean neighborhood V of y_0 such that T restricted to V is equivalent to a linear involution. Furthermore, suppose V is a Euclidean neighborhood of y_0 and $h: \partial V \to S^2 \subset E^3$ is an equivalence between $T \mid \partial V$ and $S \mid S^2$, the reflection through the origin of E^3. Then h can be extended to an equivalence between T and S.

The proofs of the above theorems depend strongly on the methods of Papakyriakopoulos [11], especially in the version of A. Shapiro and J. H. C. Whitehead [13].

2. Proof of 1.2. Let α be the generator of $\pi_1(P) \approx Z_2$. An *orientation reversing embedding* $f: P \to M$ is a one-one piecewise linear map such that $f_\#(\alpha)$ reverses orientation in M. The proof of 1.2 is similar to the proof of Dehn's lemma by Shapiro and Whitehead [13]. We shall prove the following statement, which implies 1.2:

2.1. There exists an orientation reversing embedding $f: P \to M$ if and only if there exists $\beta \in \pi_1(M)$ which reverses orientation, and such that $\beta^2 = 1$.

It is clear that given an orientation reversing embedding $f: P \to M$, then $\beta = f_\#(\alpha)$ reverses orientation (by definition) and $\beta^2 = 1$ because $\alpha^2 = 1$. It remains to find f, given β. It should be remarked that we do not prove that there exists an embedding $f: P \to M$ such that $f_\#(\alpha) = \beta$. The proof of 2.1 is broken up into several lemmas.

A map $f: P \to M$ is *canonical* if it is piecewise linear, maps each 2-simplex in a one-one fashion, and $f(P)$ has *double curves* and *triple points* (see [11, § 2]) as singularities.

To obtain canonical maps, it is convenient to use their differentiable analogues, *normal immersions*. We can assume M has a differentiable structure [5], which by [14] is unique up to diffeomorphism. A map $f: P \to M$ is an *immersion* if it is differentiable and has Jacobian of rank 2 everywhere. We say an immersion $f: P \to M$ is *normal* if whenever $x_1, \cdots, x_n$ are points of P such that $y = f(x_1) = \cdots = f(x_n)$, then the intersection of the n tangent planes to $f(P)$ at y has minimal dimension. It follows that $n \leqq 3$, and that $f(P)$ has only double curves and triple points as singularities.

It is shown in [7] that if A and B are manifolds, $\dim A < \dim B$, and A can be immersed in B, then any map $A \to B$ can be approximated by an

immersion.[3] Since P can be immersed in Euclidean 3-space (Boy's surface) P can be immersed in any 3-manifold M. In [8] it is shown that any immersion can be approximated by a normal immersion. We leave it to the reader to prove that a normal immersion can be approximated by a canonical map. (Compare this statement with [4], for example). Using these approximations and the fact that every 3-manifold has a triangulation, unique up to subdivision [9], we have proved:

LEMMA 2.2. *Any map $f: P \to M$ can be approximated (and is thus homotopic to) a canonical map.*

LEMMA 2.3. *Let $\beta \in \pi_1(M)$ be such that $\beta^2 = 1$. There exists a canonical map $f: P \to M$ such that $f_\#(\alpha) = \beta$.*

Proof. Let D be the unit disc in the complex plane and S^1 its boundary. Let $\lambda: S^1 \to M$ be a map representing β. Let $\mu: S^1 \to S^1$ be the double covering $e^{i\theta} \to e^{2i\theta}$. Then $\lambda\mu: S^1 \to M$ represents $\beta^2 = 1$, and is therefore extendable to a map $h: D \to M$. Since h sends antipodal points of S^1 into the same point, h defines a map $g: P \to M$, if we consider P to be obtained from D by identifying antipodal boundary points. It is clear that $g_\#(\alpha) = \beta$. By 2.2 there is a canonical map $f: P \to M$, homotopic to g. Thus $f_\#(\alpha) = \beta$, and 2.3 is proved.

Let $f: P \to M$ be a canonical map, and put $P_0 = f(P)$. By a *regular neighborhood* of P_0 we mean a 3-manifold V (with boundary) such that V is a sub-complex of M containing P_0, and admitting P_0 as a deformation retract. It is easy to construct regular neighborhoods (cf. [13]).

Given f and V as above, we say a covering space $p: V' \to V$ is *proper* if it is two sheeted, and $f_\#(\alpha) \in p_\#(\pi_1(V'))$.

LEMMA 2.4. *Let V be a compact non-orientable 3-manifold with non-empty boundary ∂V. If $H_1(V)$ is finite, then at least one component of ∂V is a projective plane.*

Proof. We first show that not every component of ∂V is the boundary of a 3-manifold. This is because a compact 2-manifold N which bounds some 3-manifold bounds a (possibly non-orientable) *henkelkörper* Q, in the sense of [12]. The embedding $i: N \to Q$ has the property that $i_*: H_1(N) \to H_1(Q)$ is onto. Now suppose that each component N_i of ∂V bounds a *henkelkörper* Q_i. Then the union of V and the Q_i forms a compact non-orient-

[3] The map $A \to B$ must be homotopic to an immersion. Using the results in [7], it is easy to show that any map $P \to M$ has this property.

able 3-manifold W without boundary, and because $i_*: H_1(N_i) \to H_1(Q_i)$ is onto, and $H_1(V)$ is finite, it follows that $H_1(W)$ is finite, in contradiction to [12, Satz IV, p. 206].

Thus some component of ∂V is not a boundary, and therefore is non-orientable. Let there be $s \geqq 1$ non-orientable components X_i of ∂V, whose genera (number of cross-caps) are $k_1, \cdots, k_s$, and $r \geqq 0$ orientable components Y_j whose genera are $h_1, \cdots, h_r$, if $r > 0$. We must show that some $k_i = 1$.

The Betti numbers b_n of V satisfy $b_0 = 1$, $b_1 = 0$, $b_3 = 0$, and $b_2 \geqq r$; the last relation follows from the fact that $H^3(V, \partial V) = 0$, since V is non-orientable. Let χ stand for Euler characteristic; then [12, p. 223] $\chi(V) = 2\chi(\partial V)$. Substituting in this the values $\chi(X_i) = 2 - k_i$, $\chi(Y_j) = 2 - 2h_j$, and the relations among the b_n above, we obtain, after simplifying, $s \geqq \sum_1^r h_j + \frac{1}{2}\sum_1^s k_i + 1$. Since each $h_j \geqq 0$ and each $k_i \geqq 1$, it follows that some $k_i = 1$. This prove 2.4.

LEMMA 2.5. *Let $f: P \to M$ be a canonical map and V a non-orientable regular neighborhood of $f(P)$. If V has no proper cover, then there is an orientation reversing embedding of P in V.*

Proof. The boundary ∂V of V consists of a finite number of compact 2-manifolds. We shall show that some component of ∂V is a projective plane. Let $\psi: \pi_1(V) \to H_1(V)$ be the Hurewicz homomorphism. If $H_1(V)$ were infinite, we could map $H_1(V)$ onto Z_2 such that $\psi f_\#(\alpha)$ would be in the kernel K; the covering of V corresponding to $\psi^{-1}(K)$ would then be proper, contrary to hypothesis. Therefore $H_1(V)$ is finite. By 2.4, ∂V contains a projective plane P' which is piecewise linearly embedded in M. It is clear that the embedding $P' \to V$ reverses orientation in V, and therefore also in M, and 2.5 is proved.

LEMMA 2.6. *Let $f: P \to V'$ be an orientation reversing embedding, and let $p: V' \to V$ be a double covering. There exists an orientation reversing embedding $P \to V$.*

Proof. Let $g = pf: P \to V$. By a slight deformation of f, keeping it an embedding, we may assume that g is a canonical map. Clearly, $g_\#(\alpha)$ reverses orientation in V, since $f_\#(\alpha)$ does so in V'. Since f is one-one and p is two-one, $g(P)$ has no triple points, but only double curves as self-intersections and these consist of mutually disjoint simple closed curves, piecewise linearly embedded in $g(P)$. Let C be such a double curve. Then $g^{-1}(C)$ consists of a pair (not necessarily distinct) of simple closed curves C', C'' contained in P. There are several cases to consider:

i) $C' = C''$, C' homotopically trivial

ii) $C' = C''$, C' not homotopically trivial

iii) $C' \neq C''$, both homotopically trivial

iv) $C' \neq C''$, one (say C') not homotopically trivial.

Case i) cannot occur, for both C' and C, being null-homoptopic, preserve orientation, but it is easily seen that C must reverse orientation if $C'' = C'$ (cf. [8, 3.1]). *Case* ii) is impossible, for $C' = C''$ means the map $g: C' \to C$ is a double covering, hence the homotopy class $g_\#(\alpha)$ can be represented as β^2, where β is the homotopy class of C (with some orientation). But $g_\#(\alpha)$ reverses orientation, while the square of any homotopy class preserves orientation. *Case* iii) In this situation C' and C'' bound discs and the original "cuts" (*umschaltungen*) of Dehn can be used, as described e.g. in [11]. It is easy to check that after making the cuts, the map $g_1: P \to V$ is canonical, has no triple points, fewer double curves than g, and $g_{1\#}(\alpha)$ reverses orientation. *Case* iv) is impossible, for $g_\#(\alpha) \neq 1$ (because it reverses orientation), and if C' represents α, C'' must also; but on a projective plane any two non-null-homotopic curves must intersect, contradicting the result that C' and C'' are disjoint. Thus all pairs C', C'' are as in *Case* iii), and by making a finite number of cuts, we obtain an orientation reversing embedding of P in V.

Proof of 2.1. By 2.3, there is a canonical map $g: P \to M$, such that $g_\#(\alpha) = \beta$, β reversing orientation. Let V be a regular neighborhood of $g(P)$. If V has no proper cover, then 2.5 implies 2.1. Otherwise let $p: V' \to V$ be a proper cover. Since $g_\#(\alpha) \in p_\#\pi_1(V')$ by the definition of proper cover, there is a map $g': P \to V'$ such that $pg' = g$. It is clear that g' is canonical and $g'_\#(\alpha)$ reverses orientation. Moreover, it is easily seen, as in [11, 9.1], that $g'(P)$ has fewer curves than $g(P)$. If $g'$is an embedding, 2.1 follows from 2.6. If not, we take a regular neighborhood V'' of P', a proper cover of V'' (if possible; if not, 2.1 follows from 2.5 and 2.6), and proceed inductively. Thus 2.1 is proved by 2.5, 2.6, and induction on the number of double curves of $g(P)$.

Remarks. Concerning the problem of embeddings $f: P \to M$ such that $f_\#(\alpha)$ preserves orientation, we have the following results, which are not used in the rest of the paper.

2.7. If there exists an embedding $f: P \to M$ such that $f_\#(\alpha)$ preserves orientation, then Z_2 is a free factor of $\pi_1(M)$.

Proof. Let V be a regular neighborhood of $f(P)$, where f is as in 2.6. Then $\chi(V) = \chi(f(P)) = 1$, since $f(P)$ is a deformation retract of V. Since

$2\chi(V) = \chi(\partial V)$, we have $\chi(\partial V) = 2$. Since each component of ∂V is orientable, ∂V is a 2-sphere Σ, which separates M into two parts, whose closures are V and V_1 (say), with $V \cap V_1 = \Sigma$. It follows that $\pi_1(M)$ is the free product of $\pi_1(V_1)$ and $\pi_1(V) \approx Z_2$.

In the case where f is assumed to be a *differentiable embedding*, we can choose for V a "tubular" neighborhood of $f(V)$. In this case it is easily seen, by considering the normal bundle of $f(P)$, that V is homeomorphic to the mapping cylinder of the double covering $S^2 \to P$. This proves:

2.8. A necessary and sufficient condition that there exist a differentiable embedding $f: P \to M$ such that $f_\#(\alpha)$ preserves orientation, is that M be the sum (in the sense of [12, p. 218]) of projective 3-space and another 3-manifold.

3. Proof of 1.3. We will use the following lemma.

LEMMA 3.1. *Let $R: S^2 \times I \to S^2 \times I$ be an involution without fixed points such that for $i = 0, 1, R(S^2 \times i) = S^2 \times i$ and R restricted to $S^2 \times i$ is equivalent to the antipodal map A on S^2. Then R is equivalent to $A \times e: S^2 \times I \to S^2 \times I$, where $e: I \to I$ is the identity. Furthermore, an equivalence between R and $A \times e$ on $S^2 \times 0$ can be extended to an equivalence on $S^2 \times I$.*

Proof. The orbit space $S^2 \times I \,|\, R = X$ is triangulable by Moise [9], and hence there is an induced triangulation on $S^2 \times I$ in which T is simplicial. In the rest of this section objects (maps, embeddings, etc.) will be considered from the piecewise linear point of view. Let $p: S^2 \times I \to X$ be the orbit map and let $X_i = p(S^2 \times i)$, $i = 0, 1$. It is well known [2] that an involution on S^2 without fixed points is equivalent to the antipodal map. Hence the X_i are homeomorphic to the projective plane.

We define a map of the cylinder $J = S^1 \times I$ into X as follows: Let $f_i: S^1 \times i \to X_i$, $i = 0, 1$, be embeddings which represent the non-trivial elements of $\pi_1(X_i)$. By consideration of the covering space maps $(S^2 \times i, p, X_i) \to (S^2 \times I, p, X)$, $i = 0, 1$, it is easily seen that f_0 and f_1 are homotopic in X. Thus we obtain a map $F: J \to X$ which is an extension of $f_0 + f_1$.

Since the boundary ∂J of J is embedded in ∂X, we can deform F slightly to a map $F': J \to X$ such that if $J_0 = F'(J)$ and $\partial J_0 = F'(\partial J)$, then there are no self-intersections in a neighborhood of ∂J_0 and $\partial J_0 \subset \partial X$. Then J_0 is a Dehn surface of type $(0, 2)$ in the terminology of [13]. Application of [13, 1.1] yields an embedding $G: J \to X$ which agrees with F' on a neighborhood of ∂J. (Note that 1.1 of [13] first yields a non-singular surface of type $(0, q)$, $0 < q \leqq 2$, but in our case q must equal 2 since our boundary circles are essential.)

Denote $G(J)$ by J_1 and $p^{-1}(J_1)$ by Q. Since G is essential in X, Q will be a cylinder in $S^2 \times I$ which doubly covers J_1 and $\partial Q \subset \partial(S^2 \times I)$. Furthermore Q remains invariant under R.

Let $h: S^2 \times 0 \to S^2 \times 0$ be an equivalence between R and $A \times e$, both restricted to $S^2 \times 0$, and let $K = h(Q \cap S^2 \times 0)$. Let

$$Q_0 = \{(x, y) \in S^2 \times I \mid x \in K\}.$$

Then Q_0 is invariant under $A \times e$. By the well known analogue of 3.1 in dimension 2, h can be extended to an equivalence of R and $A \times e$ on Q and Q_0. Thus we have $h: (S^2 \times 0) \cup Q \to (S^2 \times 0) \cup Q_0$. Let the components of $S^2 \times I - Q$ be denoted by A and B, the components of $S^2 \times I - Q_0$ by A_0 and B_0. Then it follows from Newman's [10] and Alexander's [1] theorems that A, B, A_0, and B_0 are 3-cells. Extend h to a homeomorphism between A and A. Finally define h on B to be $(A \times e)hR$. This h is our desired equivalence between R and $A \times e$.

We now start the proof of 1.3. As before we can assume without loss of generality that T restricted to the complement of the fixed point set is piecewise linear. The following lemma is where 1.2 is used.

LEMMA 3.2. *In every neighborhood of y_0 there is an invariant 2-sphere.*

Proof of 3.2. Let U be a given Euclidean neighborhood of y_0 in M. Choose V to be an open connected neighborhood of y_0 such that $V \subset U$ and $TV \subset U$. Let $\pi: M \to M/T$ be the orbit may, $V^* = \pi(V)$, and $V_0 = V^* - \pi(y_0)$. We claim that V_0 satisfies the conditions of 1.2. From [3] it follows that T is orientation reversing, so V_0 is non-orientable. Let $x \in V$, $x \neq y_0$, and let $b(t)$ be an arc joining x to Tx in $(V \cup TV) - y_0$. Then it is easy to see that $\beta = \{\pi b(t)\}$ satisfies the conditions of 1.2. Hence there is an embedding $j: P \to V_0$ of the projective plane in V_0. The inverse image $\pi^{-1}(j(P)) \subset U$ will be the desired 2-sphere. This proves 3.2.

Returning to the proof of 1.3, let U be as in 3.2. If N is an embedded 2-sphere in U, denote by $C(N)$ the closure of the component of $M - N$ containing y_0. Then by 3.2 choose disjoint invariant 2-sphere $V_1, V_2, V_3, \cdots$ such that $\lim(\text{diameter } V_i) = 0$, $U \supset V_1$ and for each i, $C(V_i) \supset V_{i+1}$.

Let W_i be the sphere about the origin in E^3 of radius $1/i$ and let T_0 be the reflection through the origin of E^3. Let $h: V_1 \to W_1$ be some equivalence between the restrictions of T and T_0. By 3.1 extend h to

$$C(V_1) - \text{interior } C(V_2) \to C(W_1) - \text{interior } C(W_2).$$

Here we again use Newman's and Alexander's theorems to show that

$C(V_1)$ — interior $C(V_2)$ is homeomorphic to $S^2 \times I$. Repeated applications of 3.1 yield the desired equivalence of 1.3.

4. Proof of 1.1. Let the fixed points of T be denoted by y_1 and y_2. By 3.2 let V_1 and V_2 be disjoint invariant 2-spheres about y_1 and y_2 respectively. In S^3 let $W_1 = \{x \in S^3 \mid x_1 = \frac{1}{2}\}$ and $W_2 = \{x \in S^3 \mid x_1 = -\frac{1}{2}\}$. Let $h: V_1 \to W_1$ be an equivalence between the restrictions of T and L. By 3.1, extend h to be an equivalence on the domain of $S^3 - V_1 - V_2$ not containing the fixed points. Then by 1.3 h can be extended to an equivalence on all of S^3.

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ON SPHERICAL IMAGE MAPS WHOSE JACOBIANS DO NOT CHANGE SIGN.*

By PHILIP HARTMAN and LOUIS NIRENBERG.[1]

Introduction. This paper is concerned with oriented hypersurfaces, of class C^2, of dimension n immersed in $(n+1)$-dimensional real Euclidean space E^{n+1} and with the property that the Jacobian of the spherical image mapping does not change sign. In case $n=2$, this means that the Gauss curvature does not change sign. The main result (Theorem II′) asserts that the spherical image map of a compact portion of the surface is monotone,[2] in the sense that the boundary of the image set is contained in the image of the boundary.

We shall deal mainly with hypersurfaces having simple projection on a hyperplane, i.e., admitting the representation $x^{n+1}=z(x^1,\cdots,x^n)$. The spherical image may then be adequately described in terms of the vector $p(x)=\operatorname{grad} z(x)$, where $x=(x^1,\cdots,x^n)$.

Thus a large part of the paper is concerned with mappings $p=p(x)$ of an n-dimensional x region D into n-dimensional p space, which are of class C^1 in D, of class C^0 in the closure $\bar{D}$, have the property that the vector $p(x)$ is locally a gradient (i.e., the Pfaffian $w=\sum p_k\,dx^k$ is closed, $dw=0$) and that the Jacobian of p with respect to x does not change sign. Our main result (Theorem II) asserts again that the boundary of the image of D is contained in the image $p(D')$ of the boundary D' of D.

Theorem II implies, in particular, that each component of $p(x)$ satisfies a weak maximum and minimum principle. This fact is used in Section 7

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[2] Our use of the word "montone" is derived from Lebesgue's notion [Rendiconti del Circolo Matematico di Palermo, vol. 24 (1907), p. 380] of a monotone function of two real variables. "Monotone" maps in our sense are "quasi-interior" in the sense of Whyburn [Memoirs of the American Mathematical Society, no. 1 (1950)], rather than "monotone" in his sense.

to derive an analogue of Rado's theorem [8] (see also von Neumann [6]) on saddle surfaces for n-dimensional hypersurfaces.

The basic lemmas used in proving Theorem II are given in Sections 2 and 3. In Section 3, we consider maps $p(x)$ of the type described above which have a vanishing Jacobian. For such a mapping it is shown (Theorem I) that $p(D)$ is contained in $p(D')$. The proof depends on a sharpened form and a generalization of a standard theorem on the differential geometry of torses. Theorems II and II′ are presented in Sections 4 and 5.

On the basis of Lemma 2 and its corollaries, it is shown (Theorem III in Section 6) that a complete hypersurface of constant zero curvature in $(n+1)$-(Euclidean) space is an $(n-1)$-cylinder, i.e., the hypersurface is a cylinder erected over a curve. In particular, a complete 2-surface in 3-space having zero Gaussian curvature is a cylinder; cf. Pogorelov [7].

Sections 8 and 9 which form Part II of the paper are concerned with a more detailed study of surfaces, $n=2$, having zero Gauss curvature. Consider such a surface in (x^1, x^2, x^3) space with simple projection on the (x^1, x^2) plane, i.e., which admits a representation $x^3 = z(x^1, x^2)$ on, say, a bounded convex region D in the (x^1, x^2) plane. If $x = (x^1, x^2)$ in D is the limit of points in D corresponding to nonplanar points on the surface, then, as follows from the results of Section 3 (see Section 9), through x there is a unique straight line with endpoints on the boundary of D along which the normal to the surface (or grad z) is constant. In general this will not be true for a point having a neighborhood consisting entirely of planar points. Consider the following:

Example. Let D_1 be any convex subset of D closed relative to D with boundary consisting of a subset of D' and of straight line segments with endpoints on D'. In D, define the surface as follows: $z \equiv 0$ in D_1, and the surface is a cylinder in each connected component of $D - D_1$.

Thus, in general, through a point x in D_1 there is no line segment having the properties above. However, as we prove in Section 9 (Theorem B), there always exist rays from x, going to the boundary, and along which the surface normal is constant. In Theorem A of Section 9, we also describe in general the connected subsets D_1 of a domain D which are such that the normal to the surface may be constant there.

Part I.

1. Notation. In the following, unless otherwise stated $x = (x^1, \cdots, x^n)$, $p = (p_1, \cdots, p_n)$, $q = (q^1, \cdots, q^n)$ are n vectors, D is a domain (i.e., an

open connected set, which is sometimes assumed to be bounded) in x space, and D' is its boundary. By a k-dimensional plane section π_k of D through a point x of D is meant the connected component, containing x, of the intersection of D and a k-dimensional plane through x.

Let $p = p(x)$ be of class C^1 in D. Let $J(x)$ denote the Jacobian matrix $(\partial p_i/\partial x^k)$ (so that the Jacobian itself is $\det J(x)$); $r(x)$ the rank of $J(x)$; finally $r^*(x)$ the largest integer s with the property that every neighborhood of x contains a point x^* with $r(x^*) = s$. In particular, $r(x) \leqq r^*(x)$. Let S_k denote the subset of D defined by $r^*(x) \leqq k (\leqq n)$, so that S_k is an open set.

By a gradient mapping $p = p(x)$ in D will be meant a vector valued function which is of class C^1 in D and is such that the Pfaffian

$$w = \sum p_i \, dx^i \tag{1}$$

is closed,

$$dw = 0. \tag{2}$$

Since $p = p(x)$ is of class C^1, the requirement (2) is equivalent to the condition that $J(x)$ be a symmetric matrix.

All mappings $p = p(x)$ are of class C^1 in D and, in some cases, are also assumed to be of class C^0 in the closure $\bar{D} = D \cup D'$ of D.

The image of a set A will be abbreviated by $p(A)$, its boundary by $p'(A)$.

2. Maps with Jacobian not changing sign. The mappings $p = p(x)$ in this section are defined, and of class C^1, in a bounded domain D; they are also of class C^0 in $\bar{D}$. They are *not* assumed to be gradient mappings.

LEMMA 1. *Let $p = p(x)$ be a mapping with Jacobian not changing sign. Suppose that A is a sub-domain of D satisfying*

$$p(A) \cap p(D') = 0 \text{ and } p(A) \cap p'(D) \neq 0. \tag{3}$$

Then the Jacobian $\det J$ *is identically zero in A.*

In the particular case that $D = A$, condition (3) is equivalent to

$$p(D) \cap p(D') = 0 \text{ and } p'(D) \not\subset p(D'). \tag{3'}$$

Actually, this particular case implies the general case of the lemma.

The proof of the lemma, which depends on a theorem of K. Knopp and R. Schmidt [5], was suggested by the proof of A. Douglis for the maximum principle in [3].

Proof. Since the degree of the mapping is constant on each component of the complement of $p(D')$, the first part of (3) shows that the degree of

mapping d at each point in $p(A)$ is the same; and because of the second condition there is a point $x_0 \in A$ such that $p(x_0)$ is contained in $p'(D)$ but not in $p(D')$, so that the degree of mapping at $p(x_0)$ is zero. Hence $d = 0$.

Suppose now $\det J(y) \neq 0$, say, $\det J(y) > 0$ for some point y in A. The mapping $p = p(x)$ then maps a neighborhood U of y onto a full neighborhood $p(U)$ of $p(y)$. Since the p image of the x set where $\det J(x) = 0$ is an n-dimensional zero set (Knopp-Schmidt [5]; cf. also [9]), the neighborhood $p(U)$ contains points p^* such that $\det J(x) > 0$ at every preimage of p^*. Since p^* has no preimage on D' if $U \subset A$, it has only a finite number k of preimages in D. Hence the degree d of the mapping at p^* is $k > 0$. Contradiction!

The proof of Lemma 1 yields a more general result[3] where we replace the x and p spaces by n-dimensional manifolds.

LEMMA 1'. *Let M_1, M_2 be two n-dimensional orientable manifolds of class C^2. Let M be an open connected subset of M_1 with boundary M' and compact closure $\bar{M}$. Let N be a mapping of $\bar{M}$ into M_2 which is of class C^1 in M and C^0 in $\bar{M}$. Assume, having fixed some orientations, that the Jacobian of the mapping N does not change sign. Suppose further that*

$$N(M) \cap N(M') = 0 \text{ and } N'(M) \not\subset N(M').$$

Then the Jacobian of the mapping N is identically zero.

In the remainder of this section, we present some corollaries of Lemma 1 which are not used in the rest of the paper.

COROLLARY 1. *Let D and $p = p(x)$ be as in Lemma 1. Let $v = v(p)$ be a continuous mapping of the p space E^n into a v space E^m and let $v = v(p)$ be monotone in the sense that $v'(P) \subset v(P')$ holds for every bounded domain P in p space. (E.g., let $m = n$ and $v \equiv p$ or let $m = 1$ and $v = p_1$). Let V be a closed, bounded set containing $v(p(D'))$ and having a connected complement; e.g., let V be the convex hull of $v(p(D'))$. If, for a point x_0 of D, $v(p(x_0)) \notin V$, then $\det J(x_0) = 0$. If $v = v(p)$ takes p sets of measure zero into v sets of measure zero, then* $\operatorname{vol} V \geqq \operatorname{vol} v(p(D))$.

By $\operatorname{vol} A$ is meant the m-dimensional Lebesgue measure of the (simply covered, measurable) set A in v space.

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[3] We have learned that, since this paper was completed, theorems similar to Lemma 1' have also been obtained by S. S. Chern, for $n = 2$, in "Complex analytic mappings of Riemann surfaces, I" to appear in this Journal and by S. Sternberg and R. G. Swan, for arbitrary n, in "On maps with non-negative Jacobian" to appear in the Michigan Mathematical Journal.

Remark. If $d = \operatorname{diam} v(p(D'))$ is the diameter of the set $v(p(D'))$, each of the two parts of the corollary furnishes a lower bound for d. Let V be the convex hull of $v(p(D'))$, so that $\operatorname{diam} V = d$. Let A be the open (possibly empty) subset of D where $\det J(x) \neq 0$, then by the first part, $v(p(A)) \subset V$; hence $d \geqq \operatorname{diam} v(p(A))$. When the conditions of the second part of the corollary hold, then $d \geqq 2(\operatorname{vol} v(p(D))/\omega_m)^{1/m}$, where ω_m is the volume of the m dimensional unit sphere $|v| < 1$. This inequality for d follows from a theorem of Bieberbach [1] which states that $\operatorname{vol} V \leqq \omega_m (d/2)^m$ when $d = \operatorname{diam} V$.

The inequality $d \geqq 2(\operatorname{vol} v(p(D))/\omega_m)^{1/m}$ generalizes a result of G. S. Young [11].

Proof of Corollary 1. Let $x_0 \in D$ and $v(p(x_0)) \notin V$. Suppose, if possible, that $\det J(x_0) \neq 0$. Let A be the connected component, containing x_0, of the set of points y in D such that $v(p(y)) \notin V$. Clearly, $v(p(A')) \subset V$, so that $p(A) \cap p(A') = 0$.

Since $\det J(x_0) \neq 0$, $p(x_0)$ is an interior point of $p(A)$. Let P be the connected component, containing $p(x_0)$, of the set of interior points p of $p(A)$. Since V is closed, bounded and has a connected complement, the point $v(p(x_0))$ can be joined to infinity in $E^m - V$, so that the boundary $v'(P)$ of the set $v(P)$ containing $v(p(x_0))$ is not contained in V. From the monotony of the mapping $v = v(p)$, it follows that $v'(P) \subset v(P')$, hence $v(P')$ is not contained in V. Finally, $v(P') \subset v(p'(A))$ and so, $v(p'(A))$ is not contained in V.

Hence $p(A) \cap p(A') = 0$ and $p'(A) \not\subset p(A')$. Consequently, Lemma 1 implies that $\det J$ vanishes in A, in particular, at x_0.

To prove the last part, suppose that $v(p(D)) - V$ has positive measure. Then, by our hypothesis on $v = v(p)$, the set of points $p(x)$ such that $v(p(x)) \notin V$ is not a zero set in p space. The theorem of Knopp-Schmidt [5] mentioned above implies that there is a point x with $\det J(x) \neq 0$ and $v(p(x)) \notin V$. This contradicts the first part of the corollary.

COROLLARY 2. *Let D and $p = p(x)$ be as in Lemma* 1. *Let $F(p)$ be real-valued, continuous function defined on p space and satisfying a weak maximum principle on every bounded domain (i.e., the maximum of F on the closure of the domain occurs on the boundary). Let $M = \max F(p(x))$ for $x \in D'$. If, for a point x_0 of D, $F(p(x_0)) > M$, then $\det J(x_0) = 0$.*

Proof. Clearly, $p(x_0) \notin p(D')$. Let A be the connected component, containing x_0, of the set of points y in D such that $F(p(y)) > M$. Then $F(p(x)) \leqq M$ on A', so that $p(A) \cap p(A') = 0$. Since the maximum of $F(p)$

on the boundary of every bounded p domain containing $p(A)$ is at least $F(p(x_0)) > M$, it follows that the maximum of $F(p)$ on $p'(A)$ exceeds M, so that $p'(A) \not\subset p(A')$. Consequently, by Lemma 1, $\det J \equiv 0$ on A; in particular, $\det J(x_0) = 0$.

As a consequence of Corollary 2, we have the following result of Liouville type.

COROLLARY 3. *Let $p = p(x)$ be a mapping of class C^1 of a domain D (possibly unbounded) into E^n, with Jacobian not changing sign. Suppose that there exists a non-negative, continuous function $F(p)$ defined on p space such that* (i) *$F(p)$ satisfies a weak maximum principle on every bounded domain,* (ii) *$F(p) \not\equiv 0$ on any open p set, and* (iii) *$F(p(x))$ tends to 0 as x tends to the boundary of D (or infinity). Then the Jacobian of the mapping $p(x)$ is identically zero.*

This is a more general form of a result stated in a note [11] by G. S. Young. There $D = E^n$ and it is assumed that $p(x)$ tends to a limit as $|x| \to \infty$. Corollary 3, with the choice $F(p) = |p_1 - \text{const.}|$, shows that it is sufficient to require, for example, that $p_1(x)$ tends to a limit as x tends to D'.

Note that condition (iii) can be relaxed to: (iii') there exists a sequence $D_1, D_2, \cdots$ of bounded subdomains of D such that $\bar{D}_k \subset D_{k+1}$, $D = \cup D_k$, and $m_k = \max F(p(x))$ for $x \in D_k'$ satisfies $m_k \to 0$ as $k \to \infty$.

Proof. Let x be a point where $F(p(x)) > 0$. Denote by D_x the connected component, containing x, of points y with $F(p(y)) > \frac{1}{2}F(p(x))$. By condition (iii) on F, D_x is a bounded domain and $\bar{D}_x \subset D$. By condition (i) on F, Corollary 2 is applicable on D_x, so that $\det J \equiv 0$ there; in particular, $\det J = 0$ at x. Thus the Jacobian vanishes at all points x for which $F(p(x)) > 0$.

Suppose finally that $\det J \neq 0$ at a point x where $F(p(x)) = 0$. Then $p = p(x)$ maps a neighborhood U of x onto a neighborhood $p(U)$ of $p(x)$. Since $F(p) \not\equiv 0$ for $p \in p(U)$, it follows that there is a point $y \in U$ at which $F(p(y)) > 0$, so that $\det J(y) = 0$. Since this is true for some y in every sufficiently small U, the continuity of $\det J$ implies $\det J(x) = 0$.

3. Gradient maps with zero Jacobian. Throughout the section, we consider only gradient maps, with Jacobian identically zero, of class C^1 in a domain (which may be unbounded unless otherwise stated).

LEMMA 2. *Let $p = p(x)$ be a gradient mapping with Jacobian which is identically 0. If at a point x_0 of D, $r(x_0)$ and $r^*(x_0)$ have a common value k,*

then $p(x)$ is constant on an $(n-k)$-dimensional plane section $\pi_{n-k}(x_0)$ of S_k through x_0. Furthermore, all points x near x_0 for which $p(x)=p(x_0)$ lie on π_{n-k}. Finally, $r(x)=r^*(x)=k$ for all x on π_{n-k}.

Lemma 2 is essentially a result proved by Chern and Lashof; [2], Lemma 2, p. 314. A proof of Lemma 2 is given below.

An immediate consequence of Lemma 2 is the following.

COROLLARY 1. *If $r^*(x_0)=k$ at a point x_0 of D, then $p(x)$ is constant on an $(n-k)$-dimensional plane section π_{n-k} of S_k through x_0. Also $x\in\pi_{n-k}$ implies that $r^*(x)=k$ and that either $r(x)=k$ or $r(x)<k$ according as $r(x_0)=k$ or $r(x_0)<k$.*

The first assertion of the corollary follows from that of Lemma 2 by considering a sequence of points $x_1, x_2, \cdots$ tending to x_0 such that $r^*(x_j)=r(x_j)=k$ and that $\pi_{n-k}(x_j)$ tends to a limiting position containing a section π_{n-k} as $j\to\infty$. The second assertion of the corollary follows from the last part of Lemma 2.

Note that if $p=\operatorname{grad} z$, then z is clearly a linear function on π_{n-k} (in both Lemma 2 and Corollary 1).

In general one cannot assert the uniqueness of π_{n-k} in Corollary 1. Uniqueness does hold in the following simple case.

COROLLARY 2. *If $k=1$ in Corollary 1, then the plane section π_{n-1} of S_1 through x_0 is unique—even locally.*

In order to verify this, let π_{n-1} be the $(n-1)$-section constructed above. If π'_{n-1} is another $(n-1)$-section through x_0 of a neighborhood $U\subset S_1$ of x_0 on which $p=p(x_0)$, then π'_{n-1} and $\pi_{n-1}(x_j)$ necessarily intersect at a point x_j' in U for j sufficiently large. This is based on the fact that $\pi_{n-1}(x_j)$ and π'_{n-1} are of dimension $n-1$. But by Lemma 2, $x_j'\in\pi_{n-1}(x_j)$ implies that $r^*(x_j')=r(x_j)=1$ and that there is only one $(n-1)$-section (even locally) of S_1 through x_j' on which p is constant. Contradiction. This proves Corollary 2.

As a final consequence of Lemma 2, we prove

THEOREM I. *Let $p=p(x)$ be a gradient mapping with zero Jacobian in D (so that $r(x)<n$ for all x in D). Assume that D is bounded and that p is of class C^1 in D and class C^0 in $\bar{D}$. Then $p(D)$ is contained in $p(D')$.*

Theorems I and II (below) are false if the condition that $p=p(x)$ is a *gradient* mapping is omitted, for we may take $p_1=0$ and $p_2, \cdots, p_n$ arbitrary.

Proof. Assume that the theorem is false. Let A be a component of the set of points x in D such that $p(x)$ is not in $p(D')$. Clearly A is an open set and $p(A')$ is contained in $p(D')$. Let $k = \max r(x)$ for x in A. Then $k < n$ and A is contained in S_k. There exists a point x_0 in A with $r(x_0) = k$; in particular $r(x_0) = r^*(x_0) = k$. According to Lemma 2, there is an $(n-k)$-dimensional section of S_k, hence of A, through x_0 on which $p(x) = p(x_0)$. Consequently $p(x_0)$ is contained in $p(A')$ and hence, in $p(D')$. This contradicts the definition of the set A. Thus Theorem I is proved.

We now present the

Proof of Lemma 2. (a) It will first be shown that in a small neighborhood U of x_0, the x-set $p(x) = p(x_0)$ is a unique $(n-k)$-dimensional plane section of U. Without violating the condition that $p = p(x)$ is a gradient mapping, it can be supposed that the matrix $J_k(x) = (\partial p_\alpha/\partial x_\beta)$, where $\alpha, \beta = 1, \cdots, k$ is non-singular at $x = x_0$. (For otherwise the map $x \to p(x)$ can be replaced by the map $x \to T'p(Tx)$ where T is a suitably chosen constant, non-singular matrix and T' is its transpose.) Hence the mapping $x \to q = (p_1(x), \cdots, p_k(x), x^{k+1}, \cdots, x^n)$ has a nonvanishing Jacobian at x_0, and so q can be introduced as new coordinates in a vicinity U of x_0.

By considering the Jacobian matrix $(\partial p_i/\partial q^j)$, it is seen that the assumption $r(x_0) = r^*(x_0) = k$ implies that $p_{k+1}, \cdots, p_n$ depend only on $(q^1, \cdots, q^k)$ and are independent of $(q^{k+1}, \cdots, q^n) = (x^{k+1}, \cdots, x^n)$. Hence, in U, the relations $p(x) \equiv$ const. are equivalent to $(q^1, \cdots, q^k) \equiv$ Const.

By (2), $0 = dw = \sum dp_i \wedge dx^i$. Expressing the right side of this equation in q-coordinates, and equating the coefficients of $dq^\alpha \wedge dq^i$ to zero gives

$$\partial x^\alpha/\partial q^j + \partial p_j/\partial q^\alpha = 0 \qquad \text{for } \alpha \leqq k < j.$$

Consequently, we have

$$x^\alpha = \sum_{i>k} a_i{}^\alpha x^i + b^\alpha \qquad \text{for } \alpha = 1, \cdots, k \tag{4}$$

where $-a_i{}^\alpha = \partial p_i/\partial q^\alpha$, b^α are functions of $(q^1, \cdots, q^k)$ only. Thus, in U, the hypersurface $p(x) = p(x_0)$ is an $(n-k)$-dimensional plane section of U.

(b) Let $\pi_{n-k} = \pi_{n-k}(x_0)$ be the corresponding $(n-k)$-section of S_k. It will be shown that $p(x)$ is constant on π_{n-k}. Since the change of coordinates $x \to q$ is of class C^1, the functions $a_i{}^\alpha(q^1, \cdots, q^n)$, $b^\alpha(q^1, \cdots, q^n)$ in (4) are of class C^1. Inserting (4) into $p_\alpha = p_\alpha(x)$ and differentiating with respect to $q_\beta = p_\beta$ gives

$$\delta_{\alpha\beta} = \sum_{\gamma=1}^{k} (\partial p_\alpha/\partial x^\gamma) \sum_{i>k} (x^i \partial a_i{}^\gamma/\partial q^\beta + \partial b^\gamma/\partial q^\beta) \qquad \text{for } \alpha, \beta = 1, \cdots, k.$$

Hence

$$(5)\quad 1 = \det J_k(x) \cdot \det\Big(\sum_{i>k} (x^i \partial a_i^\alpha/\partial q^\beta + \partial b^\alpha/\partial q^\beta)\Big), \quad \text{where } \alpha, \beta = 1, \cdots, k.$$

We claim that $p(x) = p(x_0)$ and $\det J_k(x) \neq 0$ on π_{n-k}. If not, there exists an arc $x = x(t)$, $0 \leqq t < 1$, on π_{n-k} starting at $x(0) = x_0$ on which either $p(x(t))$ is not constant or $\det J_k(x(t))$ vanishes somewhere. Let $t_0 < 1$ denote the least upper bound of t values for which

$$(6)\qquad p(x(s)) = p(x_0) \text{ and } \det J_k(x(s)) \neq 0$$

holds for $s \leqq t$. It is clear that $p(x(t_0)) = p(x_0)$. Furthermore the coordinate transformation $x \to q$ in part (a), hence (5), is valid in a neighborhood of each point $x(s)$ for $s < t_0$. Along $x = x(s)$, $s < t_0$, the functions $\partial a_i^\alpha/\partial q^\beta$ are independent of s. Hence (5) implies that $\det J_k(x(s))$ is bounded away from zero as $s \to t_0$. Consequently (6) holds at $s = t_0$. Since $x(t_0)$ is a point of π_{n-k}, hence of S_k, it follows that $r^*(x(t_0)) = r(x(t_0)) = k$. Applying the result of (a) at the point $x(t_0)$, it is seen that (6) holds on an s-interval larger than $[0, t_0]$. This is a contradiction, and so $p(x) \equiv p(x_0)$, $r(x) = r^*(x) = k$ on π_{n-k}.

4. Gradient maps with Jacobian not changing sign. We now present our main result.

THEOREM II. *Let $p = p(x)$ be a gradient mapping with Jacobian not changing sign in a bounded domain D, with p of class C^1 in D, C^0 in $\bar{D}$. Then $p'(D)$ is contained in $p(D')$.*

Proof. Assume the theorem to be false. Then there is a point x in D with $p(x)$ in $p'(D)$, but not in $p(D')$. Let A be the connected component, containing x, of the set of points y in D with $p(y)$ not in $p(D')$. Then $p(A') \subset p(D')$. Applying Lemma 1, we see that the Jacobian vanishes identically in A. By Theorem I, it now follows that $p(A)$ is contained in $p(A')$, and hence in $p(D')$. Contradiction!

We remark that under the conditions of the theorem, it is not true that $p(x) \in p(D')$ for every point x where the Jacobian vanishes, as the following example shows: Take $n = 2$ with coordinates (x, y), D the unit circle, and $p = \operatorname{grad}(x^2 y + y^2 x)$. Then $p \neq 0$ except at the origin, while the Jacobian is $-4(x^2 + xy + y^2) \leqq 0$ and vanishes only at the origin.

A simple consequence of Theorem II is

COROLLARY 1. *Let $p = p(x)$ satisfy the conditions of Theorem* II. *Let $F(p)$ be a continuous function defined in the p space satisfying a weak maximum principle on every bounded domain (i.e., the maximum of F on*

the closure of the domain occurs on the boundary). Then the maximum of $F(p(x))$ *on* $\bar{D}$ *occurs on* D'.

A particular case is the following:

COROLLARY 2. *Under the conditions of Theorem* II, *each component* $p_i(x)$ *of* $p(x)$ *assumes its maximum and minimum on the boundary* D'; *furthermore,* $|p(x)| = (\sum p_k^2(x))^{\frac{1}{2}}$ *assumes its maximum on* D'.

Another simple consequence of Theorem II is

COROLLARY 3. *Under the conditions of Theorem* II, *assume that the range of values of* p *on* D' *omits an open connected set* A *in* p *space. If, for some point* x_0 *in* D, $p(x_0) \in A$, *then all the* p *values in* A *are taken on by* $p(x)$ *in* D.

A particular case is

COROLLARY 4. *Under the conditions of Theorem* II, *assume that* $|p| \geqq a > 0$ *on* D', *where* a *is a constant. Then either* $|p(x)| \geqq a$ *in* D *or else* $p(x)$ *assumes in* D *all values* p *in the sphere* $|p| \leqq a$.

5. Hypersurfaces with non-simple planar projections. Theorems I and II above may be extended easily to hypersurfaces of dimension n immersed in $(n+1)$-space—without the requirement that the surface be representable in the form $x^{n+1} = z(x^1, \cdots, x^n)$.

We consider differentiably immersed compact pieces of Riemannian manifolds: Let M be a connected open subset of an oriented n-dimensional Riemannian manifold of class C^2 whose closure $\bar{M}$ is compact; denote its boundary by M'. We consider an isometric mapping $v: \bar{M} \to E^{n+1}$ with the Jacobian matrix everywhere of rank n, so that the mapping is locally one-to-one. This defines an isometric immersion of $\bar{M}$. The mapping is assumed to be of class C^2 in M and of class C^1 in $\bar{M}$.

The letters x, y will be used to denote points on $\bar{M}$ and also to denote local coordinates $x = (x^1, \cdots, x^n)$. We use $v = (v^1, \cdots, v^{n+1})$ to denote coordinates in E^{n+1}.

Let $N(x)$ be the unit normal to $v(\bar{M})$ at the point $v(x)$. Since M is oriented, a fixed choice of $N(x)$ can be made so that $N(x)$ is of class C^1 in M and of class C^0 in $\bar{M}$. Choosing fixed orientations on M and on the unit sphere in E^{n+1}, and denoting by $N(A)$ the image set under the mapping N of a set A in $\bar{M}$ and by $N'(A)$ its boundary (on the surface of the unit sphere), we can extend Theorems I, II as follows:

THEOREM I′. *Assume that the Jacobian of the mapping* $N(x)$ *is identically zero. Then* $N(M) \subset N(M')$.

THEOREM II′. *Assume that the Jacobian of the mapping* $N(x)$ *does not change sign. Then* $N'(M) \subset N(M')$.

We note that the conditions on the Jacobians are independent of the particular choice of local coordinates provided fixed orientation is maintained. Furthermore, the rank $r(x)$ of the Jacobian matrix, and hence also $r^*(x)$ as before, are well defined, independent of local coordinates.

Theorem II′ follows from Theorem I′ with the aid of Lemma 1′ in just the same way as Theorem II from Theorem I and Lemma 1.

The proof of Theorem I′ in turn follows closely that of Theorem I. Assume the theorem to be false. Let A be a component of the set of points x in M such that $N(x)$ is not in $N(M')$. A is open and $N(A') \subset N(M')$. Let k be the maximum of the rank of the Jacobian matrix of the mapping $N(x)$ in A. There exists a point x_0 in A with $r(x_0) = k$. Let U be a neighborhood of x_0 in A such that $v(U)$ has a simple projection on a hyperplane, say, the hyperplane $v^{n+1} = 0$, so that $v(U)$ admits the representation $v^{n+1} = z(v^1, \cdots, v^n)$. Denote by P the orthogonal projection on the plane $v^{n+1} = 0$. Applying Lemma 2 to the form (1) $w = dz$, we see that $v(U)$ contains a portion of π of an $(n-k)$-dimensional plane, along which the normal N to $v(U)$ is constant, and such that $P\pi$ is an $(n-k)$-section of $Pv(U)$.

About any point of $v(U)$ we can again take a neighborhood on $v(M)$ with simple projection on $v^{n+1} = 0$ and again apply Lemma 2—in order to extend the plane π. In this way one sees easily that if we draw a ray from $v(x_0)$ lying in π, and extend it, then the extension lies on the hypersurface (and along it the normal N to the hypersurface is constant) until we reach a boundary point of the hypersurface $v(A)$, i.e., a point in $v(A')$. But then it follows that $N(x_0) \in N(A') \subset N(M')$, completing the proof of the theorem.

6. On complete hypersurfaces with constant zero curvature. Consider a connected, complete, C^2, orientable, n-dimensional, Riemannian manifold M isometrically immersed in $(n+1)$-space E^{n+1} by a C^2 mapping $v = v(x)$ as in the last section. As above, x denotes either a point on M or local coordinates $x = (x^1, \cdots, x^n) \in E^n$ and $v = (v^1, \cdots, v^{n+1}) \in E^{n+1}$. Denote the immersion of M by S.

THEOREM III. *If S is of constant zero curvature, then it is an $(n-1)$-cylinder in the sense that S has a parametrization (in the large) of the form*

$$v = v(x) = \sum_{j=1}^{n-1} a_j x^j + b(x^n) \quad \text{for all } x = (x^1, \cdots, x^n),$$

where $a_1, \cdots, a_{n-1}$ are constant $(n+1)$-vectors, $b(x^n)$ is an $(n+1)$-vector of class C^2 and $a_1, \cdots, a_{n-1}, \partial b/\partial x^n$ is a set of orthonormal vectors.

This theorem depends on Corollary 2 of Lemma 2 in Section 3.

A similar result under weaker differentiability hypotheses has been stated for $n=2$ by Pogorelov [7].

Theorem III implies the following: Let the (scalar) function $z = z(x)$ be of class C^2 for all x in E^n and such that the rank $r(x)$ of the Hessian matrix $(\partial^2 z/\partial x^i \partial x^j)$ satisfies $r(x) \leqq 1$. Then, after a suitable rotation of the x space, $z(x)$ has the form $c_1 x^1 + \cdots + c_{n-1} x^{n-1} + d(x^n)$, where $c_1, \cdots, c_{n-1}$ are constants and $d(x^n)$ is a (scalar) function of x^n.

It can be supposed that M is simply connected, for otherwise it can be replaced by its universal covering manifold.

Since S is only of class C^2, the meaning of its "constant zero curvature" should be explained. Consider a piece of C^2 hypersurface $v = v(x)$, not necessarily S, in E^{n+1} as follows: let $v = v(x)$ be defined and of class C^2 on a simply connected x set D in E^n such that

$$ds^2 = |\, dv \,|^2 = g_{ik}(x)\, dx^i dx^k \tag{7}$$

is positive definite. The $g_{ik}(x)$ are of class C^1, so the the Riemann-Christoffel symbols Γ^i_{jk} of the second kind are defined and continuous. A curvature tensor can be defined extrinsically in terms of the second fundamental matrix (h_{ik}) by

$$R_{ijkl} = h_{ik} h_{lj} - h_{il} h_{kj}$$

or intrinsically by the requirement that

$$\int_{A'} \Gamma^i_{jk} dx^k = \int\int_A (R^i{}_{jkl} + \Gamma^k_{jl} \Gamma^i_{kh} - \Gamma^h_{jk} \Gamma^i_{lh})\, dx^k dx^l$$

holds for arbitrary pieces A of 2-dimensional C^1 surfaces bounded by C^1 piecewise Jordan curves A', where $\bar{A} \subset D$. These definitions are equivalent for C^2 hypersurfaces $v = v(x)$, since they are equivalent for smooth hypersurfaces.

The assumption that S has a constant zero curvature is to be interpreted to mean that if S is given locally by $v = v(x)$, then

$$h_{ik} h_{lj} - h_{il} h_{kj} \equiv 0 \qquad \text{for } i, j, k, l = 1, \cdots, n \tag{8}$$

and/or

$$\int_{A'} \Gamma^i{}_{jk}\,dx^k = \int\int_A (\Gamma^h{}_{jl}\Gamma^i{}_{kh} - \Gamma^h{}_{jk}\Gamma^i{}_{lh})\,dx^l dx^k \tag{9}$$

is an identity as explained above.

If a positive definite, C^1 metric $ds^2 = g_{ik}dx^i dx^k$, whether embedded or not, satisfies the identity (9) in A, then there exist local mappings $x \to y$ of class C^2 reducing ds^2 to the Euclidean form $ds^2 = \delta_{ik}dy^i dy^k$; [10]. This implies that the complete, simply connected manifold M can be mapped one-to-one isometrically onto an entire x space E^n. Thus, it can be supposed that M is E^n.

The identity (8) implies that the rank $r(x)$ of the Jacobian matrix of the normal image mapping of S satisfies $r(x) \leqq 1$, so that $r^*(x)$ is either 0 or 1.

Proof of Theorem III. Assume that S is not a hyperplane in E^{n+1}. Then there is a point x_0 with $r^*(x_0) = 1$. Applying Lemma 2, and its Corollary 2, to the image $v(U)$ of a neighborhood U of x_0 such that v is one-to-one in U and $v(U)$ admits a simple projection on a hyperplane, we see that through $v(x_0)$ there passes a piece τ of a unique $(n-1)$-dimensional plane on S on which N is constant. The boundary of τ is on the boundary of $v(U)$. Since the map v is isometric, the preimage of any line segment on τ, being a geodesic, is a line segment in the x space. Hence the preimage of τ is part of an $(n-1)$-dimensional hyperplane $\pi(x_0)$ in the x space and v is linear on this part of $\pi(x_0)$. On the preimage of τ, $r^*(x) \equiv 1$. Thus this argument can be repeated, replacing x_0 by any boundary point of the preimage of τ. In fact, as in the proof of Theorem I′, we see that through x_0 there is a unique hyperplane $\pi(x_0)$ on which N is constant, $v((\pi(x_0)))$ is an $(n-1)$-dimensional plane on S, $v(x)$ is linear on $\pi(x_0)$ and $r^* \equiv 1$ on $\pi(x_0)$.

It follows now that if x is another point with $r^*(x) = 1$, then the corresponding $\pi(x)$, on which $r^*(x) \equiv 1$, is parallel to $\pi(x_0)$.

Suppose now that x is a point with $r^*(x) = 0$. Let $\pi(x)$ be the hyperplane through x parallel to $\pi(x_0)$. By the preceding, we have $r^* = 0$ on $\pi(x)$. But this implies that a neighborhood on S of $v(\pi(x))$ lies on a hyperplane, so that N is constant on $\pi(x)$. Since the image of any line segment on $\pi(x)$ is a geodesic and on a hyperplane of the v space, it is a line segment. Consequently, $v(x)$ is linear on $\pi(x)$ and $v(\pi(x))$ is an $(n-1)$-dimensional plane.

Thus we have shown that through every x, there is a hyperplane $\pi(x)$ such that $v(\pi(x))$ is an $(n-1)$-dimensional plane on S along which N is constant and the hyperplanes $\pi(x)$ are parallel. After a rotation of the x

space, it can be supposed that the x^n axis is perpendicular to the $\pi(x)$. Since $v(x)$ is linear on every $\pi(x)$, it has the form

$$v(x) = \sum_{j=1}^{n-1} a_j(x^n)x^j + b(x^n),$$

where $a_1, \cdots, a_{n-1}, b$ are C^2 vectors in v space. From the isometry of the mapping, it is easily verified that $a_1, \cdots, a_n$ are constant vectors and that $a_1, \cdots, a_{n-1}$, $\partial b/\partial x^n$ are orthonormal.

7. Generalization of a theorem of Rado. As above, let E^n be the Euclidean $x = (x^1, \cdots, x^n)$ space. In this section, let E^{n+1} be the Euclidean $(x, z) = (x^1, \cdots, x^n, z)$ space, and call the z axis the vertical axis. The following terminology will be used below: A point set A_1 is said to lie above [or below] a point set A_2 in E^{n+1} if for every pair of points (x, z_1), (x, z_2) of A_1, A_2, respectively, having the same x coordinates, the inequality $z_1 \geqq z_2$ [or $z_1 \leqq z_2$] holds. By the slope of an n-dimensional hypersurface in E^{n+1} is meant the absolute value of the tangent of the angle between its normal and the z axis.

THEOREM IV. *Let D be a bounded convex domain in the x space E^n. Let $z = z(x)$ be a function of class C^2 on D and of class C^1 on $\bar{D}$. Let the Hessian matrix $(z_{ij}(x))$, where $z_{ij} = \partial^2 z/\partial x^i \partial x^j$, possess both non-negative and non-positive eigenvalues at each point x of D and satisfy*

$$\det(z_{ij}) \text{ does not change sign in } D. \tag{10}$$

(If n is even, these conditions on the Hessian hold, for example, if

$$\det(z_{ij}) \leqq 0 \text{ in } D \tag{10'}$$

holds.) Let S be the n-dimensional surface $S: z = z(x)$ in E^{n+1} and S' its boundary. Let K be a number with the property that through every point of S', there pass two n-dimensional planes with slope K such that S' lies above one and below the other. Then $|\operatorname{grad} z(x)| \leqq K$ in D.

If $n = 2$, this is implied by a theorem of Rado [8]; cf. von Neumann [6]. In Rado's theorem, the differentiability conditions and (10′) are relaxed to the assumptions that $z(x^1, x^2)$ is continuous on $\bar{D}$ and that, for arbitrary constants a_1, a_2, the function $z(x^1, x^2) - a_1x^1 - a_2x^2$ satisfies a weak maximum and minimum principle and the assertion is that $z(x)$ satisfies a Lipschitz condition with K as a Lipschitz constant.

If n is odd, the conditions on the Hessian in Theorem IV cannot be replaced by (10′). This can be seen from the example $z = R^2 - |x|^2$ on the sphere $D: |x| < R$.

Theorem IV is false for both odd and even $n(>2)$ if condition (10) is omitted (and, in fact, it is then impossible to find an a priori bound for $|\operatorname{grad} z|$). In order to obtain an example, let $\rho = (x^3)^2 + \cdots + (x^n)^2$, so that $|x|^2 = (x^1)^2 + (x^2)^2 + \rho$, and let $k(>\pi)$ be a large constant to be specified below. On the sphere $\bar{D}$: $|x| \leqq 1$, put

$$z(x) = (x^1)^2 - (x^2)^2 + (1 - |x|^2) \sin k\rho.$$

Then

$$\operatorname{grad} z = 2(x^1, -x^2, 0, \cdots, 0) - 2x \sin k\rho + 2k(1 - |x|^2)(0, 0, x^3, \cdots, x^n) \cos k\rho.$$

Let $H(x) = (z_{ij}(x))$ be the Hessian matrix of z. The 2 by 2 matrix $(z_{ij}(x))$, where $i, j = 1, 2$, in the upper left corner of $H(x)$ is a diagonal matrix with diagonal elements $2 - 2\sin k\rho$, $-2 - 2\sin k\rho$. Accordingly, $H(x)$ has an eigenvalue $\lambda \geqq 2 - 2\sin k\rho \geqq 0$ and an eigenvalue $\lambda \leqq -2 - 2\sin k\rho \leqq 0$. On D': $|x| = 1$, $z(x)$ reduces to $z(x) = (x^1)^2 - (x^2)^2$, so that there exists a constant K satisfying the assumption of Theorem IV and K can be chosen independent of k. On the other hand, if $x^1 = x^2 = 0$ and $|x|^2 = \rho = \pi/k(<1)$, then $|\operatorname{grad} z| = 2(1 - \pi/k)(\pi k)^{\frac{1}{2}}$. Hence, for such points x, $|\operatorname{grad} z| > K$ if k is sufficiently large. Thus the assertion of Theorem IV is not valid.

In the proof of Theorem IV, we shall need the following simple lemma.

LEMMA 3. Let $z = z(x)$ be a function of class C^2 on a bounded domain D in E^n and of class C^0 on $\bar{D}$. Let the Hessian matrix $(z_{ij}(x))$ have a non-positive [or non-negative] eigenvalue at every point x of D. Then $z(x)$ assumes its minimum [or maximum] value on D'.

Proof of Lemma 3. It is sufficient to verify the non-bracketed part. It can be supposed that $z \geqq 0$ in D'. It will be shown that $z \geqq 0$ on D.

Let $R > 0$ be so large that $\bar{D}$ is contained in the sphere $|x| < R$. Put $z_\epsilon = \epsilon(R^2 - |x|^2) + z(x)$ for $\epsilon > 0$. It suffices to show that $z_\epsilon \geqq 0$ in D for all $\epsilon > 0$. For the desired conclusion then follows by letting $\epsilon \to 0$.

It is clear that $z_\epsilon(x) > 0$ on D' for all $\epsilon > 0$. Suppose that, for some $\epsilon > 0$ and some $x_0 \in D$, $z_\epsilon(x_0) < 0$. Then $z_\epsilon(x)$ has a negative minimum on $\bar{D}$ which is taken at some point, say x_0, of D. Then the matrix $(z_{\epsilon ij}(x_0))$ has only non-negative eigenvalues. But $z_{\epsilon ij}(x_0) = -2\epsilon\delta_{ij} + z_{ij}(x_0)$, so that $(z_{\epsilon ij}(x_0))$ has a negative eigenvalue since $(z_{ij}(x_0))$ has a non-positive one. This contradiction proves Lemma 3.

Proof of Theorem IV. Consider the two n-dimensional planes with slope K through a point of S' such that S' is above one and below the other. An application of Lemma 3 to $z(x)$ plus a linear function shows that, not only S', but S is above one plane and below the other.

It follows that the inequality $|\operatorname{grad} z(x')| \leqq K$ holds at every point x' of D'. Since $z(x)$ is of class C^1 on $\bar{D}$, this inequality certainly holds at a point x' if D' has a tangent (unique supporting) hyperplane at x'. Since such points are dense on D', continuity considerations imply that the inequality holds at all points x' of D'. Theorem IV now follows from Corollary 2 of Theorem II (at the end of Section 4).

Part II ($n = 2$).

In this part we study more closely surfaces ($n = 2$) in 3-space with zero Gauss curvature. We shall make considerable use of Corollary 2 of Lemma 2 in Section 3.

It can be remarked that Lemma 2 and its Corollary 1 imply the following completion of a classical theorem in differential geometry (see also [7]):

Let S: $x^3 = z(x^1, x^2)$ be a surface of class C^2, defined for $(x^1)^2 + (x^2)^2 < 1$. Then the Gauss curvature of S is identically zero if and only if through every point (x^1, x^2, x^3) of S there passes a line segment on S along which the normal vector is constant.

Standard texts give a proof of the "only if" portion of this theorem only under the additional assumption that "*every or no* point of S is a flat (planar) point."

8. On parametrizations by asymptotic lines. The following is an immediate corollary of Lemma 2 and its Corollary 2.

COROLLARY 3. Let S: $x^3 = z(x^1, x^2)$ in $(x^1)^2 + (x^2)^2 < 1$ be a surface of class C^2 with zero Gauss curvature and with its nonflat points dense on $(x^1)^2 + (x^2)^2 < 1$, so that $r^* \equiv 1$. Then S has a C^0 parametrization of the form

$$x^j = a^j(u)v + b^j(u) \qquad \text{for } j = 1, 2, 3$$

where (u, v) varies over some simply connected plane domain. Here $a = (a^1(u), a^2(u), a^3(u))$ is a continuous unit vector, the vector $b = (b^1(u), b^2(u), b^3(u))$ is of class C^2 and the vector product $[a(u), b_u(u)]$ is not zero and is normal to the surface at the point (u, v).

The unit vector $a(u)$ is, of course, in the direction of the "asymptotic" line segment through the point (u, v) of S. If S has no flat points, then the above parametrization can be chosen of class C^1, but it may not be possible to choose it of class C^2, cf. [4], p. 169. On the other hand, if S contains at least one flat point, hence a line segment of such points, then

it may not be possible to choose such a parametrization of class C^1 (even if z is analytic in x^1 and x^2).

The last statement follows from the example $S: z = (x^2)^4/(2-x^1)^3$. Since $\partial z/\partial x^1 = 3(x^2)^4/(2-x^1)^4$ and $\partial z/\partial x^2 = 4(x^2)^3/(2-x^1)^3$, it follows that $\partial z/\partial x^1 = f(\partial z/\partial x^2)$, where $f(q) = 3(q/4)^{4/3}$. S has the parametrization $x^1 = v, x^2 = (u/4)^{1/3}(2-v)$, $x^3 = (u/4)^{4/3}(2-v)$, which is linear in v, and is such that the normal to S is independent of v. This parametrization is continuous but not of class C^1. An argument similar to that of [4], pp. 169-170, shows that S has no C^1 parametrization of the desired type.

As is seen from the Example in the Introduction (by choosing D_1 to be a triangle), Corollary 3 is false if the assumption $r^* \equiv 1$ is omitted. Corollary 3 has, however, a "local" analogue without this assumption.

Corollary 3'. Let S be a surface of class C^2 with zero Gauss curvature. Then every point of S has a neighborhood which admits a parametrization of the form given in Corollary 3.

Proof. We may always assume that S has the representation $x^3 = z(x)$, $x = (x^1, x^2)$, in a neighborhood of the point in question. Let the point be $(0,0,0)$. If $r = 1$ or $r^* = 0$ at the point there is nothing to prove. Thus we only consider the case that $r = 0$ and $r^* = 1$ at the origin. Suppose z is defined in $D: |x| < R$. By Lemma 2 and its Corollaries 1 and 2, we see that through every point x with $r^*(x) = 1$ there is a unique 1-section $l(x)$ of D on which grad z is constant, these line segments $l(x)$ do not intersect, and $l(x)$ depends continuously on x. We may suppose that $l(0)$ lies on the x^1-axis; then for $|x| \leqq \epsilon$, ϵ small, the lines $l(x)$ are practically parallel to the x^1-axis.

At least one of the half neighborhoods of the origin: $|x| < \epsilon$, $x^2 > 0$ or $x^2 < 0$, say $x^2 > 0$, contains a point y with $r(y) = 1$. Consider the portion $D(y)$ of $|x| \leqq \epsilon$ lying between the line $l(y)$ and the x^1-axis. We shall show that, through every point of $D(y)$, there passes a 1-section of D on which grad z = constant, that these 1-sections vary continuously, and that no two of them intersect. The set of points in $D(y)$ with $r^* = 1$ is clearly covered by such 1-sections. The remaining points, with $r^* = 0$ form an open set, and each connected component of it is bounded by two arcs on $|x| = \epsilon$ and by two 1-sections on which $r^* = 1$. Clearly the component can be covered by a continuous one parameter family of non-intersecting 1-sections on which grad z = constant (since grad z is constant in the whole component). Thus $D(y)$ may be so covered.

If the half neighbrohood $|x| < \epsilon$, $x^2 < 0$ also contains a point $\bar{y}$ with $r(\bar{y}) = 1$, we repeat this argument and obtain a covering of $D(\bar{y})$ which together with the preceding yield a covering of a whole neighborhood of

$x = 0$; the corollary then follows immediately. If the half neighborhood $|x| < \epsilon$, $x^2 < 0$ contains only points with $r^* = 0$, then we may cover it by lines $x^2 =$ constant (on which, of course, grad z is constant). This covering with the preceding yield again a covering of a whole neighborhood of $x = 0$.

9. The sets $N(x) =$ constant. We shall consider again surfaces $x^3 = z(x) = z(x^1, x^2)$ with simple projection on a domain D of the $x = (x^1, x^2)$ plane, and shall make some attempt to characterize the sets of D on which N, the normal—or what is the same, $p =$ grad z—can be constant. Thus in the following $p(x)$ is a gradient mapping, with zero Jacobian, defined in a domain D (not necessarily bounded) in the plane; p is of class C^1 in D and of class C^0 in $\bar{D}$.

Definition. For any point x_0 of D, let $C(x_0)$ be the arcwise connected component, containing x_0, of the set of points x for which $p(x) = p(x_0)$.

The following assertions concerning $C(x_0)$ will be proved:

THEOREM A. (a) *If $x \in D$ is a boundary point of $C(x_0)$, then there is a 1-section of D through x belonging to the boundary of $C(x_0)$. Thus, the boundary of $C(x_0)$ consists of 1-sections of D and a subset of the boundary D' of D.* (b) *If D is simply connected and contains a closed line segment $[x_1 x_2]$, where the endpoints are points of $C(x_0)$, then $C(x_0)$ contains the line segment $[x_1 x_2]$. In particular, if D is convex, so is $C(x_0)$.*

It is easy to see that (b) is false if D is not simply connected.

Also, in space, (b) is false. In fact let $n = 3$, write (x, y, z) for x, and let $p = \text{grad}(x + z^2(1 - y^2)^2) = (1, -4yz^2(1 - y^2), 2z(1 - y^2)^2)$, so that the Jacobian vanishes identically. The set $p = (1, 0, 0)$ consists of the three planes $z = 0$, and $y = \pm 1$. Since this set is arcwise connected, it is $C(0, 0, 0)$, but it is not convex.

Proof of Theorem A. Since a point x in D which is in $C(x_0)$ belongs also to the boundary of $C(x_0)$ if and only if $r^*(x) = 1$, part (a) of the theorem follows from Lemma 2 and its Corollary 1. In order to prove part (b) suppose that the segment $[x_1 x_2]$ contains a point x not in $C(x_0)$. It may be supposed that $p(x) \neq p(x_0)$, $r^*(x) = 1$. By Lemma 2, p is constant on a 1-section S of D through x, which necessarily has x_1, x_2 on opposite sides. In case D is convex, this contradicts the connectedness of $C(x_0)$ and proves (b) in this case. In general, the connectedness of $C(x_0)$ implies that not both endpoints of S are at infinity.

Let $\mathscr{C}$ be an arc in $C(x_0)$ joining x_1 and x_2. It can be supposed that $\mathscr{C}$ does not meet the open segment $[x_1 x_2]$, for otherwise x_1 can be replaced by

the last intersection x_1', going from x_1 to x_2, of $\mathscr{C}$ and the segment $[x_1 x]$, and x_2 by the first intersection after x_1' of $\mathscr{C}$ and the segment $[x x_2]$. By a similar argument it can be supposed that $\mathscr{C}$ has no self intersections. Thus $\mathscr{C}$ together with $[x_1 x_2]$ forms a simple closed curve in D, which by the Jordan curve theorem, has one endpoint of S inside it and the other outside. Since the endpoints of S are boundary points of D this contradicts the assumption that D is simply connected, and proves (b).

Remark. In case D is a bounded convex domain, then a subset D_1 is a possible region $C(x_0)$ if and only if it is of the form in the Example of the Introduction.

We now prove the existence of rays on which p is constant. First a *definition*: Let $x_0 \in D$. By a ray from x_0 in D we mean the connected portion, containing x_0, of the intersection of D with a half infinite straight line having x_0 as endpoint.

THEOREM B. *If $x_0 \in D$, then there exist at least two rays emanating from x_0 on which p is constant. If D has the property that it contains no infinite (in both directions) straight line and if $r^*(x_0) = 0$, then there are at least three such rays from x_0.*

Proof. In view of Lemma 2 and its Corollary 1, it can be supposed that $r^*(x_0) = 0$, so that x_0 is an interior point of $C(x_0)$. Furthermore, we may suppose that D is star shaped about x_0, and so, simply connected, otherwise, we restrict ourselves to the star-shaped domain made up of all rays from x_0 in D.

Consider a ray from x_0 on which p is not constant. (If none such exists, we have nothing more to prove.) Moving along the ray from x_0 we come to a first point x_1 of D belonging to the boundary of $C(x_0)$, so that $r^*(x_1) = 1$. Let $\pi(x_1)$ be the 1-section through x_1 of Theorem A(a) on which p is constant. It follows from Theorem A(b) that the straight segment joining x_0 to any point of $\pi(x_1)$ belongs to $C(x_0)$, so that p is constant on it. But then it follows that p is constant on the straight segments joining x_0 to the endpoints of $\pi(x_1)$ (which may be at infinity). These segments are rays of D from x_0, so that we have shown the existence of two rays having the required property.

Suppose now that D contains no infinite straight line. Then the triangle made up by the straight segments from x_0 to points of $\pi(x_1)$ above, on which p is constant, has an angle at the vertex x_0 smaller than π. Let us now consider a ray from x_0 in D which does not belong to the triangle and on which p is not constant. Repeating the argument above we obtain a 1-section

$\pi(x_1')$ such that p is constant on the straight segment joining x_0 to any point of it. But the sections $\pi(x_1)$, $\pi(x_1')$ can have at most one endpoint in common. Thus we have three distinct rays from x_0 on which p is constant.

Remark 1. That there need not exist more than three such rays can be seen from the Example in the Introduction, where for D_1 we take a triangle with vertices on the boundary of D. Then D_1 serves as a set $C(x_0)$ for any point x_0 inside it, and from any such point, the only rays on which p is constant are those to the vertices of D_1.

Remark 2. The proof of Theorem B shows for *any* D that if, from x_0 in D, there are two non-collinear rays on which p is constant, then there is a third.

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COLLINEATION GROUPS OF NON-DESARGUESIAN PLANES, I.*

The Hall Veblen-Wedderburn systems

By D. R. HUGHES.[1]

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1. Introduction. In the modern study of projective planes, a problem of considerable interest is that of determining the collineation groups of the known non-Desarguesian planes. For the group reveals much about the structure of the plane, and interesting problems of group theory can be involved. Perhaps the most thorough investigation of this sort to date is due to Zappa ([9]); he determined at least part (and possibly all) of the collineation group of a typical "Hughes plane," showing among other things that the group is necessarily not solvable. Essentially, this was because the collineation group contains a subgroup isomorphic to a three dimensional projective group. In this paper we shall show that the collineation group of a Hall Veblen-Wedderburn plane contains the two dimensional general linear group over an appropriate field; this group is non-solvable with certain exceptions. The only exceptional case that interests us (when the field under consideration is $GF(3)$) is shown to give rise to a non-solvable group for other reasons. Also, the whole collineation group of a finite Hall Veblen-Wedderburn plane is determined.

In Section 2 the Hall Veblen-Wedderburn systems are defined. Sections 3 and 4 are concerned, respectively, with existence and uniqueness theorems on collineations of the planes. In Section 5 we find conditions that non-isomorphic systems shall coordinatize the same plane; in particular, it is shown that two finite Hall Veblen-Wedderburn planes of the same order are isomorphic. Section 6 is concerned with the exceptional plane of order 9, which is the same as the problem of the field $GF(3)$ mentioned above.

Since this research was carried out, it has been shown by Albert that the planes coordinatized by "twisted fields" have solvable collineation groups, and still more recently the author has shown that all the other known finite non-associative division rings give planes with solvable collineation groups. It is not clear what conclusions, if any, should be drawn from these facts.

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versations about this paper; in particular, he pointed out the truth of Lemma 5.1 and Theorem 6.1, thus filling what would have otherwise been gaps in the treatment.

2. The Hall Veblen-Wedderburn systems. For a background on the notion of the most general coordinatizing system for projective planes, see [2, 3, 7] and in particular [3] for the scheme used here. We shall only be interested in a special type of coordinate system and we restrict attention to that system.

Suppose R is a set containing at least the two distinct elements 0 (zero) and 1 (one), and possessing two binary operations, addition and multiplication. We let R^* denote the set of non-zero elements of R. If the following are satisfied, then R is called a *left Veblen-Wedderburn system* (left V-W system, or merely V-W system, for short):

(a) $(R, +)$ is a group with "identity" zero;

(b) $(R^*, \cdot)$ is a loop with identity 1;

(c) $a(b+c) = ab + ac$, for all a, b, c in R;

(d) if a, b, c in R, $a \neq b$, then there is a unique x in R such that $ax = bx + c$.

It has been shown that then $(R, +)$ is necessarily abelian, and if R is finite, $(R +)$ is even elementary abelian, so that the order of R, (i.e., number of elements) is a prime power. Furthermore, if R is finite, then (d) is redundant. (See [7] for proofs.) If we wish to emphasize the operations of R, we write $(R, +, \cdot)$.

If R is a V-W system, a projective plane π is constructed from R as follows: the points of π are the symbols (x, y), (x), (∞) for all x, y in R, and where "∞" is a symbol not in R; the lines of π are the symbols $[m, k]$, $[\infty, (k, 0)]$, L_∞, for all m, k in R. The incidence rules of π are: (x, y) is on $[m, k]$ if $mx + y = k$, and (x, y) is on $[\infty, (k, 0)]$ if $x = k$; (x) is on $[m, k]$ if $x = m$, and (x) is on L_∞; (∞) is on L_∞ and on $[\infty, (k, 0)]$. (Again, see [2, 3, 7] for more details.) A *collineation* of π (indeed, of any projective plane) is a one-to-one mapping of points onto points, lines onto lines, which preserves incidence.

If π is a projective plane coordinatized by a V-W system (e.g., as above), then it is known ([2, 7]) that every coordinate system for π with the same choice of L_∞ (the line at infinity) is also a V-W system (not neces-

sarily isomorphic), and if R is not an alternative division ring, then no other choice of the line at infinity will lead to a coordinate ring which is a V-W system. Hence in particular, if R is a V-W system but not an alternative division ring, no collineation of π can move the line L_∞. (We recall that finite alternative division rings are fields.)

Now we proceed to the construction of a class of V-W systems. Let $K \neq GF(2)$ be any field over which there exist irreducible quadratics, and let $f(z) = z^2 - pz - q$ such an irreducible quadratic over K.[2] Define R to be the set of all elements $\lambda a + b$, where a, b are in K (and λ is a new symbol), with an addition in R given by:

$$(1) \qquad (\lambda a + b) + (\lambda c + d) = \lambda(a + c) + (b + d).$$

Define multiplication in R by:

$$(2) \quad (\lambda \cdot 0 + b)(\lambda c + d) = \lambda(bc) + (bd),$$

$$(3) \quad \text{if } a \neq 0, \text{ then } (\lambda a + b)(\lambda c + d) = \lambda(ad - bc + pc) + bd - a^{-1}c(b^2 - pb - q).$$

Then R is a left V-W system, never satisfies the right distributive law, and never satisfies the associative law for multiplication except in the single instance that $K = GF(3)$ and $p = 0$, $q = -1$. (Proofs are in [2, 7].) Such a system will be called a *Hall* V-W system, and the projective plane coordinatized by it a Hall V-W plane. Since the set of elements of the form $\lambda \cdot 0 + b$ forms a subfield isomorphic to K, we will identify this subset with K.

It is not at all hard to prove, using (2), (3) above, that the following rules hold in R:

(4) if a is in R, then $ax = xa$ for all x in R if and only if a is in K;

(5) if a is in R, then $(xy)a = x(ya)$ for all x, y in R if and only if a is in K, or $K = GF(3)$, $p = 0$, $q = -1$, in which case a can be any element of R;

(6) if x is in R, but not in K, then $x^2 = px + q$.

Indeed, Hall's original definition ([2]) of these systems was made by means of (1), (4), (5), (6), whence one can show (2), (3) as consequences. In view of (4) and (5), we call K the *center* of R.

The automorphism group of R has been determined in [4]; we will give

[2] In case $K = GF(2)$, the resulting system is $GF(4)$; for certain purposes, $GF(4)$ does indeed behave like a Hall V-W system, but the exceptions are numerous enough to warrant excluding it from consideration here.

it after giving a larger group of "autotopisms" which turns out to have considerable importance. Let a, b, c, d be in K, with $ad - bc \neq 0$, and define $S = S(a, b, c, d)$ to be the following mapping of R (clearly one-to-one and onto):

$$(\lambda x + y)S = \lambda(ax + by) + (cx + dy).$$

Then (see [6]), S satisfies:

(7) $(x + y)S = xS + yS$, for all x, y in R;

(8) $(xy)S = (xs)S(yS)$, for all x, y in R, where s is the element defined by $sS = 1$.

Furthermore:

(9) if x, y, u, v are in R, $y \neq 0$, $v \neq 0$, x not in Ky, u not in Kv, then there is a unique mapping $S = S(a, b, c, d)$ such that $xS = u$, $yS = v$.

If $\mathfrak{B}$ is the set of all mappings $S(b, 0, a, 1)$, with $b \neq 0$, then the automorphism group of R is the direct product of the group $\mathfrak{B}$ and the group of automorphisms of K which fix the elements p and q. The group $\mathfrak{B}$ is transitive on the set of elements of R which are not in K, and thus, if it is convenient, we can choose λ, say, as a "typical" element of R which is not in K.

3. Existence of collineations. Now let R be the Hall V-W system constructed over the field K using the irreducible quadratic $f(z) = z^2 - pz - q$, and let π be the projective plane coordinatized by R. We shall proceed to exhibit various collineations of π.

(1) The translation group $\mathfrak{T}$; this group is possessed by all V-W planes. For each pair a, b in R, define $\tau = \tau(a, b)$ by:

$$\begin{aligned} \tau: (x, y) &\to (x + a, y + b) & [m, k] &\to [m, k + ma + b] \\ (x) &\to (x) & [\infty, (k, 0)] &\to [\infty, (k + a, 0)] \\ (\infty) &\to (\infty) & L_\infty &\to L_\infty. \end{aligned}$$

(2) The automorphism group $\mathfrak{A}$. For each automorphism α of K which fixes p and q, define α on R by $(\lambda x + y)\alpha = \lambda(x\alpha) + (y\alpha)$, and then define α on π by:

$$\begin{aligned} \alpha: (x, y) &\to (x\alpha, y\alpha) & [m, k] &\to [m\alpha, k\alpha] \\ (x) &\to (x\alpha) & [\infty, (k, 0)] &\to [\infty, (k\alpha, 0)] \\ (\infty) &\to (\infty) & L_\infty &\to L_\infty. \end{aligned}$$

(3) The multiplication group $\mathfrak{M}$. If a is in K^*, let $\mu = \mu(a)$ be defined by:

$$\begin{aligned} \mu: (x,y) &\to (xa, ya) & [m,k] &\to [m, ka] \\ (x) &\to (x) & [\infty, (k,0)] &\to [\infty, (ka,0)] \\ (\infty) &\to (\infty) & L_\infty &\to L_\infty. \end{aligned}$$

(4) The autotopism group $\mathfrak{S}$. For each mapping S, as defined in Section 2, define $\sigma = \sigma(S)$ as follows:

$$\begin{aligned} \sigma: (x,y) &\to (xS, yS) & [m,k] &\to [(ms)S, kS] \\ (x) &\to ((xs)S) & [\infty, (k,0)] &\to [\infty, (kS,0)] \\ (\infty) &\to (\infty) & L_\infty &\to L_\infty. \end{aligned}$$

Here, as in Section 2, s is the element defined by $sS = 1$. Note that $\mathfrak{M} \subseteq \mathfrak{S}$, since if $S = S(a,0,0,a)$, then $\sigma(S) = \mu(a)$.

(5) The linear group $\mathfrak{L}$. For each choice of a, b in K, not both zero, define $\zeta = \zeta(a,b)$ by:

$\zeta: (x,y) \to ((pa+b)x + ay, qax + by)$

$(x) \to (x)$, if x is not in K

$(x) \to (-(xb - qa)(xa - pa - b)^{-1})$, if x is in K, but $xa \neq pa + b$.

$(p + ba^{-1}) \to (\infty)$, if $a \neq 0$

$(\infty) \to (-ba^{-1})$, if $a \neq 0$

$(\infty) \to (\infty)$, if $a = 0$

$[m,k] \to [m, mka + bk]$, if m is not in K

$[m,k] \to [-(mb - qa)(ma - pa - b)^{-1},$
$\quad -k(b^2 + pab - qa^2)(ma - pa - b)^{-1}],$
$\quad$ if m is in K, but $ma \neq pa + b$

$[p + ba^{-1}, k] \to [\infty, (ka, 0)]$, if $a \neq 0$

$[\infty, (k,0)] \to [-ba^{-1}, -ka^{-1}(b^2 + pab - qa^2)]$, if $a \neq 0$

$[\infty, (k,0)] \to [\infty, (kb, 0)]$, if $a = 0$

$L_\infty \to L_\infty$.

Note also that $\mathfrak{M} \subseteq \mathfrak{L}$, since $\zeta(0,a) = \mu(a)$.

(6) The involution δ defined by:

$$\begin{aligned} \delta: (x,y) &\to (-x, px + y) & [m,k] &\to [-m + p, k] \\ (x) &\to (-x + p) & [\infty, (k,0)] &\to [\infty, (-k, 0)] \\ (\infty) &\to (\infty) & L_\infty &\to L_\infty. \end{aligned}$$

Excepting the cases of the linear group $\mathfrak{L}$ and the involution δ, it is obvious that the above mappings are collineations. We will sketch the proofs for these cases before proceeding to a discussion of the various groups.

Consider $\zeta(a, b)$ in $\mathfrak{L}$. Suppose (x, y) is on $[m, k]$, whence $mx + y = k$. Then if m is not in K, we must show that

$$m[(pa + b)x + ay] + qax + by = m(mx + y)a + b(mx + y).$$

Straightforward simplification reduces the above equation to:

$$mx(pa + b) + mya + qax + by = [m(mx) + my]a + mxb + yb,$$

which reduces to

$$mxp + qx = m(mx)$$

if $a \neq 0$, and if $a = 0$ we are done. Now we can assume that $m = \lambda$ and $x = \lambda u + v$, where u, v are in K. Then we have:

$$\lambda(\lambda u + v)p + (\lambda u + v)q = \lambda[\lambda(\lambda u + v)],$$

which is easy to verify, using the fact that $\lambda^2 = \lambda p + q$.

On the other hand, if m is in K, $ma \neq pa + b$, then the equation to be checked is:

$$-(mb - qa)[(pa + b)x + ay](ma - pa - b)^{-1} + qax + by$$
$$= -(mx + y)(b^2 + pab - qa^2)(ma - pa - b)^{-1}.$$

Since only x and y might not be in K, this equation is easily simplified and verified.

The remaining possiblities are either trivial or similarly straightforward, and so $\mathfrak{L}$ is a collineation group. Note that the one-to-one-ness of $\zeta(a, b)$ is assured by the fact that $b^2 + pab - qa^2$ is zero only if $a = b = 0$; for if $a \neq 0$, say, then $b^2 + pab - a^2 = a^2(c^2 - pc - q) = a^2 f(c) \neq 0$, where $c = -ba^{-1}$. The verification that δ is a collineation is similar, while the assertion than δ is an involution (i.e., $\delta^2 =$ identity) is immediate.

We remark that the subset of π consisting of L_∞, (∞), and all points and lines all of whose coordinates are in K is a subplane of π; we will call this subplane π'. Clearly π' is Desarguesian, with K as its coordinate field (see [7]). We shall let π'_∞ denote the set of points of π' which are on L_∞, and for convenience, $L_\infty - \pi'_\infty$ will denote the set consisting of the remaining points on L_∞. By the *finite points* of π (or π') we will mean the set of points of π (or π') which are not on L_∞. Finally, $\mathfrak{G}$ will denote the group of all collineations of π.

THEOREM 3.1. *The groups of collineations given above have the following properties:*

(a) $\mathfrak{T}$ *is transitive on the finite points of* π *and is normal in* $\mathfrak{G}$.

(b) $\mathfrak{S}$ *fixes every point of* π'_∞ *and is transitive on* $L_\infty - \pi'_\infty$.

(c) $\mathfrak{Q}$ *fixes every point of* $L_\infty - \pi'_\infty$ *and is transitive on* π'_∞.

(d) $\mathfrak{M} = \mathfrak{S} \cap \mathfrak{Q}$.

(e) *If* $\mathfrak{V} = \{\mathfrak{Q}, \delta\}$ = *group generated by* $\mathfrak{Q}$ *and* δ, *then* $[\mathfrak{V}:\mathfrak{Q}] \leqq 2$.[3]

(f) $\mathfrak{S}$ *is isomorphic to the group of all non-singular two-rowed square matrices over* K, *and hence is non-solvable if* K *has more than three elements.*

(g) $\mathfrak{Q}$ *is isomorphic to the multiplicative group of the quadratic extension field* $K[w]$, *where* $w^2 = pw + q$.

Proof. (a) is well-known, see for instance [7]. The proof of (b) will be found in [6]. (c) is immediate from the definition of $\mathfrak{Q}$, while (d) and (e) are straightforward. Noting the fact that the mapping S acts on the two-dimensional vector space R (over K) exactly as a linear transformation, (f) is clear. Finally, (g) is evident when we set up the correspondences:

$$\zeta(a, b) \leftrightarrow \begin{bmatrix} pa + b & a \\ qa & b \end{bmatrix} \leftrightarrow aw + b \text{ in } K[w].$$

We shall see later that when $K \neq GF(3)$, then $\mathfrak{S}$ and $\mathfrak{Q}$ are both normal in $\mathfrak{G}$; it can be demonstrated that under any circumstances each normalizes the other. Indeed, it will be one of our aims to show that we have given essentially all of $\mathfrak{G}$, except in the case that $K = GF(3)$ or K is infinite. Specifically, we shall show that $\mathfrak{G}/\mathfrak{T} \cong \mathfrak{V}\mathfrak{S}\mathfrak{A}$. In order to achieve this, we must show that every collineation of π maps π'_∞ into itself, and so although π' is not an invariant subplane, its intersection with L_∞ is invariant.

4. Uniqueness of collineations. We shall examine here the possibility that other collineations than those generated by the groups of Section 3 can exist. From [6], π possesses a collineation group fixing π'_∞ pointwise, also fixing $(0, 0)$, and transitive on the points $(x, 0)$, $x \neq 0$, of the x-axis (in fact, $\mathfrak{S}$ is the necessary group); certainly $\mathfrak{T}$ is transitive on the finite points of π. So if two different coordinatizations of π exist, using the same points (∞), (0), (1), such that one of the two (at least) is a Hall V-W system, then we

[3] $[\mathfrak{V}:\mathfrak{Q}] = 2$ excepting possibly in an infinite field of characteristic two, where δ will be the identity if $p = 0$.

can assume that both use the same points $(0,0)$ and $(1,0)$ (and hence the same points $(1,1)$ and $(0,1)$ also). Thus if the set R of symbols is assumed to be the same for both systems, it is easy to show (see [2, 7] for instance) that R has a permutation ϕ, where $0\phi = 0$, $1\phi = 1$, such that the two systems $(R, +, \cdot)$ and $(R, \oplus, *)$ are related by:

$$(x \oplus y)\phi = x\phi + y\phi, \qquad (x * y)\phi = (x\phi)(y\phi).$$

That is, the two systems are isomorphic.

If $(R, +, \cdot)$ is a Hall V-W system over the field K (= its center), then $(R, \oplus, *)$ is a Hall V-W system over the field $K' = K\phi^{-1}$. If K_1 is the prime field of K, then $K_1 = K_1\phi$, so that the set of symbols in K_1 make up the prime field of K' also. (We shall see that for our purposes we can even take $K' = K$.)

Suppose π is finite, with order n; then L_∞ contains $n+1$ points, while π'_∞ contains n^2+1 points. Since n^2+1 never divides $n+1$, it follows that if a collineation of π moves π'_∞, then there are some pairs of images of π'_∞ which are not equal but have a non-empty intersection. So if a collineation moves π'_∞, there is a collineation moving π'_∞ but keeping the point (∞) fixed (since $\mathfrak{G}$ is transitive on π'_∞). This argument breaks down in the infinite case, and so we restrict attention to finite Hall V-W systems excepting insofar as some of the results have restricted validity for infinite systems. Thus we now consider the situation that (∞) is fixed; our conclusions will be valid for collineations of infinite planes which fix π'_∞.

So we suppose that a collineation of π moves (0) to (r), (1) to (s), and fixes (∞) and $(0,0)$. We will investigate a coordinate system for π (using primes for the new coordinates) which has $(\infty)' = (\infty)$, $(0)' = (r)$, $(1)' = (s)$, $(0,0)' = (0,0)$. From the remarks above, we can complete the coordinatization in any convenient fashion, and must arrive at a Hall V-W system isomorphic to $(R, +, \cdot)$. Let $(x,0)' = (x, -rx)$. Then since $(1)'$, $(0,y)'$, and $(y,0)'$ are collinear, it follows that $(0,y)' = (0, sy - ry)$. Since $(x,y)'$ is collinear first with $(\infty)'$ and $(x,0)'$ and second with $(0)'$ and $(0,y)'$, one easily shows that $(x,y)' = (x, sy - ry - rx)$. Finally, $(m)'$ is collinear with $(1,0)'$ and $(0,m)'$, so $(m)' = (sm - rm + r)$.

It is easy to show that in the new system, which we will call $(R, \oplus, *)$, the addition is unchanged: $x \oplus y = x + y$. The multiplication is as follows: if $m * x = k$, then $(m)' = (sm - rm + r)$ and $(x,0)' = (x, -rx)$ are collinear with $(0,k)' = (0, sk - rk)$. So:

$$(sm - rm + r)x - rx = sk - rk. \tag{1}$$

LEMMA 4.1. *The set of elements of K is precisely the center of the system $(R, \oplus, *)$.*

Proof. It is easy to show that the only elements of a Hall V-W system which distribute on *both* sides are the elements of its center. Now let x be in K, in (1). Then we have $smx - rmx = sk - rk$, or $s(mx - k) = r(mx - k)$, whence $m * x = k = mx$, since $s \neq r$. So $(u \oplus v) * x = (u \oplus v)x = x(u \oplus v)$ $= x(u + v) = xu + xv = ux + vx = u * x \oplus v * x$, for any u, v in R. Hence K must be in the center of $(R, \oplus, *)$. But $(R, \oplus)$ is the same two-dimensional vector space over K as it is over its own center, since $+$ and $\oplus$ are the same operation. Thus it follows that K is precisely the center of $(R, \oplus, *)$.

THEOREM 4.1. *If K is finite and $K \neq GF(3)$, then no collineation of π moves a point of π'_∞ outside of π'_∞.*

Proof. It suffices to show that r and s must be in K. We let m be in K, whence $k = m * x = mx$, from Lemma 4.1.

Case 1. Suppose r is in K, but s not in K; we take $s = \lambda$, without loss of generality. If $x = \lambda a + b$, and m is in K, then (1) becomes:

$$(2) \quad (\lambda m + r - rm)(\lambda a + b) - r(\lambda a + b) = \lambda(\lambda ma + mb) - r(\lambda ma + mb).$$

If $m \neq 0$, this simplifies to:

$$(3) \quad \lambda[a(p + rm - 2r) + bm] - m^{-1}a[(r - m)^2 - p(r - rm) - q] - brm = \lambda[a(pm - rm) + bm] + qam - brm.$$

Since a and b are arbitrary, (3) leads to:

$$(4) \quad \lambda[p - pm + 2rm - 2r] - m^{-1}[(r - rm)^2 - p(r - rm) - q] - qm = 0.$$

Equating coefficients to zero, we have:

$$(5) \quad p(1 - m) - 2r(1 - m) = 0.$$

$$(6) \quad qm^2 + (r - m)^2 - p(r - m) - q = 0.$$

If $m \neq 1$, then (5) yields $p = 2r$; substituting this in (6), we have:

$$(7) \quad m^2(q + r^2) = q + r^2.$$

Thus $m^2 = 1$ or $q = -r^2$. But if $q = -r^2$, then $f(z) = z^2 - pz - q$ $= (z - r)^2$, which is impossible, as $f(z)$ is irreducible, and so $m^2 = 1$. So the only non-zero values in K that m can take are $+1$ and -1; hence $K = GF(3)$.

Case 2. Suppose s is in K and r is not in K; we assume $r = \lambda$. As above, let $x = \lambda a + b$. Then if m is in K, but $m \neq 1$, equation (1) becomes:

$$(8) \qquad \lambda(pm - 2sm) - q + qm - (1-m)^{-1}(s^2m^2 - psm - q) = 0.$$

From $pm - 2sm = 0$, we deduce $p = 2s$, by letting m be non-zero. From the second coefficient in (8), this yields:

$$(9) \qquad m(s^2 + q) - 2(s^2 + q) = 0.$$

As before, $s^2 + q = 0$ implies $f(z) = (z - s)^2$, which is a contradiction, so we must have $m = 2$. So if m is in K, it must have one of the values 0, 1, 2, and thus again $K = GF(3)$.

Case 3. Suppose neither s nor r are in K, and $r = \lambda$, $s = \lambda a + b$, where $a \neq 0, 1$. As before, let m be in K, and $x = \lambda u + v$. First let $m = (1-a)^{-1}$, whence (1) becomes:

$$(10) \qquad \lambda(ap + 2b - p) + aq + a^{-1}(b^2 - pb - q) = 0.$$

Equating coefficients to zero, we have $p = 2b(1-a)^{-1}$ from the λ term, and then from the second term:

$$(11) \qquad q(1-a)(a^2 - 1) = b^2(a + 1).$$

If $a = -1$, then (11) is valid. If $a \neq -1$, then (11) yields $q = -b^2(1-a)^{-2}$, and then $f(z) = [z - b(1-a)^{-1}]^2$, which is impossible. So $a = -1$, and also $p = b$.

Now we assume that $m \neq (1-a)^{-1} = 2^{-1}$, $p = b$ in (1), obtaining:

$$(12) \qquad (1-2m)^{-1}(p^2m^2 - p^2m - q) + q = 2qm,$$

and on simplification, this is:

$$(13) \qquad m^2(p^2 + 4q) - m(p^2 + 4q) = 0.$$

If K does not have characteristic two, then $p^2 + 4q$ is the discriminant of $f(z)$, hence is not zero, so (13) implies that $m = 0$ or $m = 1$; thus once again we are led to $K = GF(3)$. If $2 = 0$ in K, then $a = -1 = 1$ and this violates the assumption of Case 3.

Case 4. Suppose $r = \lambda$, $s = \lambda + b$, where $b \neq 0$ of necessity. As before, we take m in K, $x = \lambda u + v$, and equation (1) becomes:

$$(14) \qquad b^2(m^2 - m) = 0.$$

Since $b \neq 0$, this implies $m = 0$ or $m = 1$, so $K = GF(2)$, which is impossible.

Thus if $K \neq GF(3)$, we have shown that r and s must be in K, so we are done with the proof of the theorem.

THEOREM 4.2. *If K is finite, $K \neq GF(3)$, then $\mathfrak{G} = \mathfrak{VSAT}$, and both $\mathfrak{S}$ and $\mathfrak{Q}$ are normal in $\mathfrak{G}/\mathfrak{T}$.*

Proof. Suppose γ is a collineation of π which fixes (∞), $(0,0)$, and maps π'_∞ into itself. We can represent γ as below, where T, U, V, W, Z are appropriate mappings of R, or $R \times R$, onto R.

$$\begin{aligned}\gamma\colon (x,y) &\to ((x,y)T, (x,y)U)\\ (x) &\to (xV)\\ [m,k] &\to [mV, (m,k)W]\\ [\infty, (k,0)] &\to [\infty, (kZ,0)].\end{aligned}$$

If $x = k$, then $((x,y)T, (x,y)U)$ is on $[\infty, (kZ,0)]$, so $(x,y)T = xZ$. Hence we redefine T so that $xT = xZ = (x,y)T$. If (x,y) is on $[m,k]$, then $mx + y = k$ and:

$$mV \cdot xT + (x,y)U = (m, mx+y)W. \tag{15}$$

If $x = 0$ in (15), then $(m,y)W = (0,y)U$, since $0T = 0$. So we redefine W so that $kW = (m,k)W$. Then (15) is:

$$mV \cdot xT + (x,y)U = (mx+y)W. \tag{16}$$

Letting $m = 0$ and noting that $0V = r$, where r is in K, (16) yields $(x,y)U = yW - r(xT)$, so (16) becomes:

$$mV \cdot xT + yW - r \cdot xT = (mx+y)W. \tag{17}$$

Now let $m = 1$, $y = 0$ in (17), and note that $1V = s$, where s is in K; we have then $xW = s \cdot xT - r \cdot xT = (s-r)xT$. Then (17) is:

$$mV \cdot xT - r \cdot xT + (s-r)yT = (s-r)[(mx+y)T]. \tag{18}$$

If $m = 1$ in (18), then we have $(s-r)xT + (s-r)yT = (s-r)[(x+y)T]$, so $(x+y)T = xT + yT$, since $s \neq r$. So we have:

$$mV \cdot xT - r \cdot xT = (s-r)[(mx)T]. \tag{19}$$

Noting that T is one-to-one and onto, let t in R satisfy $tT = 1$. Letting $x = t$ in (19), we find:

$$mV = (s-r)[(mt)T] + r, \tag{20}$$

$$[(s-r) \cdot (mt)T + r] \cdot xT - r \cdot xT = (s-r) \cdot (mx)T. \tag{21}$$

Now if m is in K, then mV is in K, since γ maps π'_∞ onto π'_∞. So $(mt)T$ is in K; define $m\phi = (mt)T$, for all m in K. Then letting $x = yt$, where y is in K, and letting m be in K, (21) implies $m\phi \cdot y\phi = (my)\phi$. Furthermore, if x and y are in K, then $(x+y)\phi = [(x+y)t]T = (xt+yt)T = x\phi + y\phi$. So ϕ is an automorphism of K.

Now let $\lambda T = \lambda a + b$, $1T = \lambda c + d$. Letting m be in K, $x = \lambda$, (21) becomes:

$$[(s-r)\cdot m\phi + r]\cdot \lambda T - r\cdot \lambda T = (s-r)\cdot (m\lambda)T,$$

or:

$$(m\lambda)T = m\phi\cdot\lambda T = \lambda a\cdot m\phi + b\cdot m\phi. \tag{22}$$

Similarly, letting m be in K and $x = 1$, (21) yields:

$$mT = m\phi\cdot 1T = \lambda c\cdot m\phi + d\cdot m\phi. \tag{23}$$

Hence, since T is an additive map, we have:

$$(\lambda x + y)T = \lambda(a\cdot x\phi + c\cdot y\phi) + (b\cdot x\phi + d\cdot y\phi). \tag{24}$$

But then T is associated with a collineation in $\mathfrak{CA}$ (see Section 3) and so we can assume that T is the identity. Then the collineation γ becomes:

$$\begin{aligned}
&\gamma: (x,y) \to (x, (s-r)y - rx)\\
&\quad (x) \to ((s-r)x + r)\\
&\quad [m,k] \to [(s-r)m + r, (s-r)k]\\
&\quad [\infty, (k,0)] \to [\infty, (k,0)].
\end{aligned}$$

From this we have immediately:

$$[(s-r)m + r]x - rx = (s-r)mx. \tag{25}$$

Now if m is in K, (25) is easily seen to be satisfied; if we take $m = \lambda$, then straightforward computation from (25) leads to:

$$p - 2r = p(s-r) \tag{26}$$

$$-(r^2 - pr - q) = q(s-r)^2. \tag{27}$$

Now suppose $2 \neq 0$ in K. Then squaring both sides of (26) and dividing by (27), we are led to:

$$(r^2 - pr)(p^2 + 4q) = 0. \tag{28}$$

As above, $p^2 + 4q$ is the discriminant of $f(z)$, so is not zero, and thus (28) implies $r = 0$ or $r = p$. From this one easily deduces that $r = 0$ and $s = 1$,

or $r=p$ and $s=p-1$. But if $r=0$, $s=1$, then γ is the identity map, while if $r=p$, $s=p-1$, then γ is the involution δ (see Section 3).

On the other hand, if $2=0$ in K, then (26) and (27) become:

$$p=p(s+r), \qquad r^2+pr+q=q(s+r)^2. \tag{29}$$

If $p\neq 0$, then $s+r=1$, and then from the second equation of (29) we have $r=0$, $s=1$, or $r=p$, $s=p+1=p-1$. So again γ is the identity or $\gamma=\delta$.

If $p=0$, then $f(z)=z^2+q$ is irreducible; since every element is a square if K is finite, K must be infinite and q a non-square. But if $p=0$, the second equation of (29) is $r^2+q=q(s+r)^2$, so $r^2=q(1+s+r)^2$. Since q is a non-square, this implies $1+s+r=0$ and $r=0$, hence $s=1$. Thus γ is the identity.

Up to a multiple by an element of $\mathfrak{Q}$, every collineation can be assumed to fix the point (∞), so we have finished the first part of the theorem. The fact that $\mathfrak{S}$ and $\mathfrak{Q}$ are normal in $\mathfrak{G}/\mathfrak{T}$ follows directly from an easy computation, since each normalizes the other and $\mathfrak{A}$ normalizes both of them. Alternatively, note that $\mathfrak{S}$ and $\mathfrak{Q}$ are the sugroups of $\mathfrak{V}\mathfrak{S}\mathfrak{T}$ fixing pointwise the two transitive constituents on L_∞, so they must normalize each other, and it is easy to see that $\mathfrak{A}$ normalizes both.

Note that we have shown that if γ maps π'_∞ onto itself, then γ is in $\mathfrak{V}\mathfrak{S}\mathfrak{A}\mathfrak{T}$, even in the case $K=GF(3)$, or K infinite.

5. Different Hall *V*-*W* systems. In this section we will investigate the conditions under which non-isomorphic Hall V-W systems, over the same center K, lead to isomorphic planes; it is easy to see that the use of different irreducible quadratics will always lead to non-isomorphic systems. In this section we will also allow K to be infinite.

We return to equation (1) of Section 4.

$$(sm-rm+r)x-rx=sk-rk, \tag{1}$$

where $k=m*x$, as in Section4. Letting s, r be in K in (1), we can write:

$$m*x=(s-r)^{-1}[((s-r)m+r)x]-(s-r)^{-1}rx. \tag{2}$$

As demonstrated in Section 4, this multiplication satisfies properties (4) and (5) of Section 2, so we must consider property (6). Thus we let $m=x=\lambda$ in (2). Multiplying out, this yields:

$$\lambda*\lambda=\lambda(s-r)^{-1}(p-2r)-(s-r)^{-2}(r^2-pr-q).$$

Hence if we let $p_1 = (s-r)^{-1}(p-2r)$, $q_1 = -(s-r)^{-2}(r^2-pr-q)$, we have:

$$\lambda * \lambda = \lambda p_1 + q_1. \tag{3}$$

Since (3) does not depend on the choice of m $(=x)$, property (6) of Section 2 follows immediately. I.e., $(R, \oplus, *)$ is a Hall V-W system over the field K as center, using $f_1(z) = z^2 - p_1 z - q_1$ as the irreducible quadratic ($f_1(z)$ must be irreducible, for otherwise $(R, \oplus, *)$ would have zero-divisors; see [7]). Since the group $\mathfrak{L}$ moves (∞) anywhere in π'_∞, we have proved:

THEOREM 5.1. *Any choice of* $(\infty)'$, $(0)'$, $(1)'$ *in* π'_∞ *will lead to a Hall V-W system.*

Now we investigate a converse problem: which Hall V-W systems can be obtained in this fashion? We can always assume that $(\infty)' = (\infty)$, and so we must decide if s and r can be so chosen that

$$p_1(s-r) = p-2r, \tag{4}$$

$$q_1(s-r)^2 = -(r^2-pr-q), \tag{5}$$

for a given p, q, p_1, q_1, with the proviso that $f(z) = z^2 - pz - q$ and $f_1(z) = z^2 - p_1 z - q_1$ are irreducible over K.

When is $f(z)$, say, irreducible? If $2 \neq 0$ in K, then if and only if $p^2 + 4q$ is not a square, as is well-known. But if $2 = 0$, we have no such condition. We will develop one of a strikingly similar sort, one which appears to be vaguely known, but does not seem to be in the literature. Let K be a field (finite or infinite) of characteristic two and let $P(K)$ be the set of elements in K which can be written as $x^2 + x$, for some x in K.

LEMMA 5.1. *If* $2 = 0$ *in* K, *then* $P = P(K)$ *is an additive subgroup of* $(K, +)$, *and if* K *is finite,* P *contains exactly half of the elements of* K. *Furthermore, the quadratic* $az^2 + bz + c$ (*where* $a \neq 0$) *is irreducible over* K *if and only if* (i) $b \neq 0$ *and* ca/b^2 *is not in* P, *or* (ii) K *is imperfect,* $b = 0$, *and* c/a *is not a square in* K. *Finally, the irreducible quadratics* $az^2 + bz + c$ *and* $a_1z^2 + b_1z + c_1$ *define the same quadratic extension fields of* K *if and only if* (iii) $b \neq 0$, $b_1 \neq 0$, *and* $ca/b^2 + c_1a_1/b_1^2$ *is in* P, *or* (iv) K *is imperfect,* $b = b_1 = 0$ *and* $x^2c/a + y^2c_1/a_1$ *is a square for some non-zero* x *and* y *in* K.

Proof. The first sentence is easy to prove. The second part depends on noting that $az^2 + bz + c = (b^2/a)[(az/b)^2 + (az/b) + (ca/b^2)]$ if $b \neq 0$, so $az^2 + bz + c = 0$ for some z in K if and only if $ca/b^2 = (az/b)^2 + (az/b)$.

I.e., ca/b^2 is in P. The rest of the proof is straightforward computation, which we omit.

Note that if we only consider perfect fields of characteristic two, then conditions (ii) and (iv) are superfluous, and the analogy with the discriminant becomes clearer. In general, if $2=0$ in K and if $b\neq 0$, we shall call ac/b^2 the discriminant of the expression az^2+bz+c.

Now we return to (4) and (5). Squaring (4) and dividing by (5), we have:

$$r^2(p_1^2+4q_1)-pr(p_1^2+4q_1)+p^2q_1-p_1^2q=0. \tag{6}$$

Thinking of (6) as a quadratic equation for r, its discriminant, if $2\neq 0$, is $\Delta=p_1^2\delta\delta_1$, where $\delta=p^2+4q$, $\delta_1=p_1^2+4q_1$. Then Δ is a square if and only if $p_1=0$ or $\delta\delta_1$ is a square. But $\delta\delta_1$ is a square if and only if the quadratic extension fields defined over K by $f(z)$ and $f_1(z)$ are the same; if $p_1\neq 0$ we can solve for s if we can solve (6) for r. On the other hand, if $p_1=0$, then (4) implies $2r=p$, and then (5) becomes:

$$(s-r)^2=(r^2+q)/q_1=\delta/\delta_1, \tag{7}$$

and we can solve for s if and only if δ/δ_1 is a square. But this is equivalent to $\delta\delta_1$ being a square, and again, we are reduced to the case that $f(z)$ and $f_1(z)$ define the same quadratic extension field of K.

Now if $2=0$ in K, (6) becomes:

$$r^2p_1^2+pp_1^2r+p^2q_1+p_1^2q=0. \tag{8}$$

If $p\neq 0$, $p_1\neq 0$, then the discriminant of (8) is quickly computed to be $\Delta=\delta+\delta_1$, where $\delta=q/p^2$, $\delta_1=q_1/p_1^2$. So (8) can be solved for r if and only if Δ is in $P(K)$, which is to say, if and only if $f(z)$ and $f_1(z)$ define the same quadratic extension field of K; and of course we can solve for s if we can solve for r.

If $p=0$, $p_1\neq 0$, then (8) becomes $r^2+q=0$, but this is presumably without solutions, since $f(z)$ is irreducible, and so (8) cannot be solved. But also the extension fields defined by $f(z)$ and $f_1(z)$ are not isomorphic in this case. If $p\neq 0$, $p_1=0$, then (8) becomes $p^2q_1=0$, which is impossible, so we have a similar situation.

Finally, if $p=p_1=0$, then (8) gives no information, so we return to (5), which yields:

$$q_1(s+r)^2=r^2+q. \tag{9}$$

This is solvable for s and r if and only if $(r^2+q)/q_1$ is a square. If such

an s and r exist, then letting $x = s + r$, $y = 1$, $z = r$ gives $x^2q_1 + y^2q = z^2$. Conversely, if $x^2q_1 + y^2q = z^2$ for non-zero x, y, z in K, then we let $s + r = x/y$, $r = z/y$. So, utilizing Lemma 5.1, (9) is solvable if and only if $f(z)$ and $f_1(z)$ define the same quadratic extension field of K.

THEOREM 5.2. *If $f(z)$ and $f_1(z)$ are irreducible quadratics over K, then the Hall V-W planes defined by $f(z)$ and $f_1(z)$ are isomorphic if $f(z)$ and $f_1(z)$ have zeros in the same quadratic extension field of K.*[4]

COROLLARY. *Two finite Hall V-W systems of the same order define isomorphic projective planes.*

The corollary is of course immediate since there is only one finite field of any given order. Theorem 5.2 probably has a converse, but the detailed investigation of moving the point (∞) is necessary, and this is extremely tedious.

6. The case $K = GF(3)$. Throughout this section we restrict attention to $K = GF(3)$; in view of the corollary to Theorem 5.2, we need consider only one Hall V-W system. So we let $p = 0$, $q = -1$, and then the V-W system R is even a near-field; i.e., the multiplication is associative. We shall determine what collineations, besides those of Section 3, π possesses. To begin with, let $\mathfrak{N}$ be the group defined as follows. For each t in R, $t \neq 0$, define θ_t by:

$$\theta_t: (x, y) \to (x, ty) \qquad [m, k] \to [tm, tk]$$
$$(x) \to (tx) \qquad [\infty, (k, 0)] \to [\infty, (k, 0)],$$

where (∞) and L_∞ are fixed. Then $\mathfrak{N}$ is a collineation group, and taken with the groups of Section 3, it is clear that $\mathfrak{G}$ is transitive on the points of L_∞.

André has discovered ([1]) that for each point P on L_∞ there is a uniquely defined point $P' \neq P$ such that $(P')' = P$, and such that any collineation of π must map a pair (P, P') onto a pair (Q, Q'). André's results are not stated in this fashion, but the properties of the "zulässige Punktepaare" of [1] are easily seen to force this conclusion. Thus the pairs (P, P') are systems of imprimitivity on L_∞.

[4] In some recent papers (Rivista Mat. Univ. Parma, vol. 8 (1957); Convegno Internazionale Reticoli e Geometrie Proiettive, (1957)) and in a paper to appear in Rendiconti dell'Accademia Nazionale dei Lincei, Gianfranco Panella has investigated the conditions under which two Hall V-W systems give isomorphic planes, and has obtained results cast in somewhat different form than ours.

LEMMA 6.1. *If* $P = (\infty)$, *then* $P' = (0)$, *and if* $P = (m) \neq (0)$, *then* $P' = (-m)$.

The proof of Lemma 6.1 can be found in [1], although it is easy to use Section 3 to give a proof, using the fact that the pairs exist.

Now let $\mathfrak{G}_1$ be the collineation group of π generated by the groups of Section 3 and the group $\mathfrak{N}$. Let $\bar{\mathfrak{G}}_1$ be the restriction of $\mathfrak{G}_1$ (as a permutation group) to the imprimitive systems (P, P') on L_∞, so that $\bar{\mathfrak{G}}_1$ is a permutation group of degree 5. We note that $\mathfrak{G}_1$ is a subgroup of $\mathfrak{G}$ and that $\bar{\mathfrak{G}}_1$ is a homomorphic image of $\mathfrak{G}_1$, so if we show that $\bar{\mathfrak{G}}_1$ is non-solvable, the same result will hold for $\mathfrak{G}$.

LEMMA 6.2. $\bar{\mathfrak{G}}_1$ *contains a transposition.*

Proof. The collineation $\zeta(1,1)$ in $\mathfrak{Q}$ fixes every point (x), x not in K, and interchanges the pair $((\infty), (0))$ with the pair $((1), (-1))$. So in $\bar{\mathfrak{G}}_1$ $\zeta(1,1)$ induces a transposition.

LEMMA 6.3. $\bar{\mathfrak{G}}_1$ *is doubly transitive on its five symbols.*

Proof. The collineations of $\mathfrak{N}$ fix the pair $((\infty), (0))$ and are transitive on the remaining points of L_∞; hence they are certainly transitive on the pairs $(P, P') \neq ((\infty), (0))$. Since $\mathfrak{G}_1$ is transitive on the pairs (P, P'), this implies that $\bar{\mathfrak{G}}_1$ is transitive on its five symbols and the subgroup fixing one symbol is transitive on the remaining four; thus $\mathfrak{G}_1$ is doubly transitive.

THEOREM 6.1. $\mathfrak{G}$ *is non-solvable.*

Proof. As remarked above, it suffices to show that $\bar{\mathfrak{G}}_1$ is non-solvable. But $\bar{\mathfrak{G}}_1$ is doubly transitive and contains a transposition, so it is isomorphic to the symmetric group on five symbols, and thus is non-solvable.

We have now shown that the collineation group of any Hall V-W plane is non-solvable, even though we have not determined all of the group (perhaps) for the infinite planes of the class.

Let us study the group $\mathfrak{G}$ a little more for this case $K = GF(3)$. We suppose γ is in $\mathfrak{G}$; up to a multiple by an element in $\mathfrak{N}\mathfrak{Q}$, we can assume that γ fixes (∞), and from André's result ([1]), a collineation which fixes (∞) also fixes (0). Again up to a multiple by an element of $\mathfrak{N}$, we can assume that (1) is fixed (and thus (-1) is fixed); because of $\mathfrak{T}$, $(0,0)$ can be assumed to be fixed. Then from the remark at the end of Theorem 4.2, our collineation is already contained in $\mathfrak{S}$ (for $K = GF(3)$, $\mathfrak{A}$ clearly has order one). Thus:

THEOREM 6.2. $\mathfrak{G}_1 = \mathfrak{G}$.

Indeed, one can also show that the order of $\mathfrak{G}$ is $10 \cdot 81 \cdot 48 \cdot 8$, since the subgroup fixing $(0,0)$ and (∞) is $\mathfrak{SN}$, of order $48 \cdot 8$; the involution δ of Section 3 is equal to the element θ_{-1} of $\mathfrak{N}$. (And $\mathfrak{SN}$ is a group, for $\theta_t \sigma(S) = \sigma(S)\theta_{(ts)s}$.) The determination of the group $\mathfrak{G}$, as well as its order, was already carried out by André in [1], by the way.

Finally, it is easy to show that for any Hall V-W plane, the collineation group is doubly transitive on finite points if and only if it is transitive on the points of L_∞. Hence for the finite planes of the class, the plane of order nine is the only one whose group is doubly transitive on finite points.

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A PARTITION FUNCTION WITH SOME CONGRUENCE CONDITION.*

By Shô Iseki.

Introduction. It is well known that the number $p(n)$ of unrestricted partitions of a positive integer n can be expressed by a convergent series (see [9]).

In this paper we shall be concerned with the number $p(n;a,M)$ of partitions of n into positive summands congruent to $\pm a$ modulo M, where a, M are integers with $M \geqq 2$.

This partition function $p(n;a,M)$ has been treated for certain special values of M; namely, $M=2$ by Hua [2], $M=5$ by Lehner [5], $M=6$ by Niven [8], and $M=p$ (p a prime >3) by Livingood [6]; a convergent series representation of $p(n;a,M)$ being obtained in each case.

The main object of the present paper is to derive a convergent series for $p(n;a,M)$ in which M assumes *general* values.

We may suppose without loss of generality that $0<a<M$, $(a,M)=1$; for in the case where $a=0$ and $M \mid n$, the partition function in question reduces trivially to $p(n/M)$, while in the case where $d=(a,M)>1$ and $d \mid n$, it reduces to $p(n/d;a/d,M/d)$. Further we remark that the case $M=2$ (partitions into odd summands) is equivalent to the case $M=4$, i.e., $p(n;1,2)=p(n;1,4)$. Consequently it suffices to consider the case $M \geqq 3$.

We apply the Hardy-Ramanujan method with modifications due to Rademacher [10]. In order to utilize this method, however, it is necessary to solve the following two subsidiary problems:

The first problem is to find a suitable transformation equation for the generating function of $p(n;a,M)$. The usual method of contour integration seems to be rather complicated in our case. However, we can show, in a simple way, the existence of the transformation equation, the proof of which being based on a certain functional equation recently obtained by the author [3]. This will be done in Section 1 of this paper.

The second is to get a non-trivial estimate for a certain exponential sum which is introduced as a consequence of the transformation equation. We shall

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discuss this problem in Section 2. The analysis used is somewhat elaborate, but we can reduce the sum to a generalized Kloosterman sum, for which suitable estimates are known. To achieve this result we follow partly a method due to Lehner [5]. But Lehner's original method seems to be effective only for the case $(6, M) = 1$. We therefore have to make a partial revision of his method to secure the case $(6, M) > 1$.

We shall derive the desired series expansion in Section 3, by making use of the results of Sections 1 and 2.

The final section is devoted to the investigation of some asymptotic properties of $p(n; a, M)$ which are deducible from the series representation.

The author wishes to express his gratitude to Professor Joseph Lehner for kind information on the literature.

1. The transformation equation. The generating function of $p(n; a, M)$ is found to be, for $M \geqq 3$,

$$F(x; a, M) = 1 + \sum_{n=1}^{\infty} p(n; a, M) x^n = \prod_{m=0}^{\infty} (1 - x^{mM+a})^{-1} \ (1 - x^{mM+M-a})^{-1},$$

where x is a complex variable with $|x| < 1$.

Now let h, k be coprime integers with $k \geqq 1$. Denote by D and K the g. c. d. and the l. c. m. of k and M respectively. Put $k = k_1 D$, $M = m_1 D$, so that $(k_1, m_1) = 1$, and choose any integers γ, δ satisfying

$$\gamma k_1 - \delta m_1 = 1. \tag{1.1}$$

Let, then, H be any fixed solution of the congruence [1]

$$hH \equiv \delta \pmod{k}. \tag{1.2}$$

Furthermore we write

$$x = \exp(2\pi i h/k - 2\pi z/k), \qquad x' = \exp(2\pi i H/k - 2\pi/Kz), \tag{1.3}$$

where z is a complex variable with $\Re(z) > 0$. Define

$$F(x'; b, D, \rho) = \prod_{m=0}^{\infty} (1 - \rho x'^{mD+b})^{-1} \ (1 - \rho^* x'^{mD+D-b})^{-1} \tag{1.4}$$

with the notations

$$b = ha - D[ha/D], \quad \rho = \exp(-2\pi i \gamma a/M), \quad \rho^* = \exp(2\pi i \gamma a/M), \tag{1.5}$$

[1] In the case $(k, M) = M$, we have $m_1 = 1$. Hence we may choose $\gamma = 0$, $\delta = -1$ in (1.1). Then (1.2) becomes $hH \equiv -1 \pmod{k}$. This congruence has been used by Lehner [5] and Livingood [6].

in which $[t]$ denotes the greatest integer not exceeding t. (Note that $b \equiv ha \pmod{D}$, $0 \leqq b < D$, $(b, D) = 1$; $\rho^* = \rho^{-1}$.)

Then there exists a transformation equation which connects $F(x; a, M)$ with $F(x'; b, D, \rho)$, and therefore exhibits the asymptotic behavior of $F(x; a, M)$ near its singularity at each rational point on the unit circle.

THEOREM 1.[2] *Let*

$$\begin{aligned} &\omega(h,k) = \exp\{2\pi i\sigma(h,k)\}, \\ &\sigma(h,k) = \sum_{\mu} (\mu/K - \tfrac{1}{2})(h\mu/k - [h\mu/k] - \tfrac{1}{2}), \end{aligned} \tag{1.6}$$

where μ runs over the integers $a, M+a, 2M+a, \cdots, (k_1-1)M+a$; and set

$$A = 6a^2 - 6Ma + M^2, \qquad B = 6b^2 - 6Db + D^2. \tag{1.7}$$

Then, if $M \geqq 3$, we have the transformation equation

$$F(x; a, M) = \omega(h,k)\exp\{(\pi/6Mk)(B/z - Az)\}F(x'; b, D, \rho). \tag{1.8}$$

To prove Theorem 1 we need the following lemma.

LEMMA 1. *If μ runs over the integers*

$$a, M+a, 2M+a, \cdots, (k_1-1)M+a,$$

then $\mu^* = h\mu - k[h\mu/k]$ runs, in some order, over the integers

$$b, D+b, 2D+b, \cdots, (k_1-1)D+b.$$

Proof. First, it is easy to see that the assumed values of μ are k_1 integers which are mutually incongruent modulo k, and hence, observing that $\mu^* \equiv h\mu \pmod{k}$ and $(h,k) = 1$, it follows that the values of μ^* are distinct k_1 integers of the interval $0 \leqq \mu^* < k$.

Next, the facts that $\mu^* \equiv h\mu \pmod{k}$ and that $\mu \equiv a \pmod{M}$ together yield $\mu^* \equiv ha \pmod{D}$, so that $\mu^* \equiv b \pmod{D}$ since $ha \equiv b \pmod{D}$.

On the other hand, any integer μ^* satisfying both $0 \leqq \mu^* < k$ and $\mu^* \equiv b \pmod{D}$ must be one of the integers $b, D+b, 2D+b, \cdots, (k_1-1)D+b$ since $0 \leqq b < D$ and $k_1 D = k$. But these are exactly k_1 integers. Thus our lemma is proved.

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[2] The case $D = 1$ of Theorem 1 can also be expressed in a different form, see [4], Theorem 2.

Proof of Theorem 1. We apply the following functional equation ([3], p. 654, Theorem 1):

$$\begin{aligned}(1.9)\qquad &\sum_{l=0}^{\infty}\{\lambda((l+\alpha)z-i\beta)+\lambda((l+1-\alpha)z+i\beta)\}+\pi z(\alpha^2-\alpha+\tfrac{1}{6})\\ &=\sum_{l=0}^{\infty}\{\lambda((l+\beta)/z+i\alpha)+\lambda((l+1-\beta)/z-i\alpha)\}\\ &\qquad+(\pi/z)(\beta^2-\beta+\tfrac{1}{6})+2\pi i(\alpha-\tfrac{1}{2})(\beta-\tfrac{1}{2}),\end{aligned}$$

where $0<\alpha<1$, $0\leqq\beta\leqq 1$ (or $0\leqq\alpha\leqq 1$, $0<\beta<1$), $\Re(z)>0$, and $\lambda(t)$ denotes $-\log(1-e^{-2\pi t})$, the logarithm having its principal value.

Let us put

$$(1.10)\qquad \alpha=\mu/K,\qquad \beta=\mu^*/k,$$

where μ, μ^* are those of Lemma 1, so that we have $0<\alpha<1$, $0\leqq\beta<1$, since $0<\mu<k_1M=K$ and $0\leqq\mu^*<k$. Substituting (1.10) into (1.9) and replacing z by m_1z, the left member of (1.9) becomes

$$\begin{aligned}(1.11)\qquad &\sum_{l=0}^{\infty}\{\lambda((l+\mu/K)m_1z-i\mu^*/k)+\lambda((l+1-\mu/K)m_1z+i\mu^*/k)\}\\ &\qquad+\pi m_1z\{(\mu/K)^2-\mu/K+\tfrac{1}{6}\}\\ &=\sum_{l=0}^{\infty}\{\lambda((Kl+\mu)(z-ih)/k)+\lambda((Kl+K-\mu)(z-ih)/k)\}\\ &\qquad+\pi m_1z\{(\mu/K)^2-\mu/K+\tfrac{1}{6}\},\end{aligned}$$

since $K=m_1k$ and $\mu^*/k\equiv h\mu/k \pmod 1$; while the right member of (1.9) becomes

$$\begin{aligned}(1.12)\qquad &\sum_{l=0}^{\infty}\{\lambda((l+\mu^*/k)/m_1z+i\mu/K)+\lambda((l+1-\mu^*/k)/m_1z-i\mu/K)\}\\ &\qquad+(\pi/m_1z)\{(\mu^*/k)^2-\mu^*/k+\tfrac{1}{6}\}+2\pi i(\mu/K-\tfrac{1}{2})(\mu^*/k-\tfrac{1}{2})\\ &=\sum_{l=0}^{\infty}\{\lambda((kl+\mu^*)/Kz+i\mu/K)+\lambda((kl+k-\mu^*)/Kz-i\mu/K)\}\\ &\qquad+(\pi/m_1z)\{(\mu^*/k)^2-\mu^*/k+\tfrac{1}{6}\}+2\pi i(\mu/K-\tfrac{1}{2})(\mu^*/k-\tfrac{1}{2}).\end{aligned}$$

Now, it follows from the congruence $\mu^*\equiv h\mu \pmod k$ and (1.2) that $H\mu^*\equiv hH\mu\equiv\delta\mu \pmod k$, and so

$$(1.13)\qquad m_1H\mu^*\equiv m_1\delta\mu \pmod{m_1k=K}.$$

On the other hand, $\mu\equiv a \pmod M$ implies

$$(1.14)\qquad \gamma k_1\mu\equiv\gamma k_1a \pmod{k_1M=K}.$$

It follows from (1.13), (1.14) and (1.1) that

$$\mu\equiv -m_1H\mu^*+\gamma k_1a \pmod K$$

and hence, noting that $K = m_1 k = k_1 M$, we have

$$\begin{aligned} \mu/K &\equiv -m_1 H\mu^*/K + \gamma k_1 a/K \pmod 1 \\ &= -H\mu^*/k + \gamma a/M. \end{aligned} \tag{1.15}$$

Therefore the right member of (1.12) transforms into the form

$$\begin{aligned} &\sum_{l=0}^{\infty} \{\lambda((kl+\mu^*)(1/Kz - iH/k) + i\gamma a/M) \\ &\quad + \lambda((kl+k-\mu^*)(1/Kz - iH/k) - i\gamma a/M)\} \\ &\quad + (\pi/m_1 z)\{(\mu^*/k)^2 - \mu^*/k + \tfrac{1}{6}\} + 2\pi i(\mu/K - \tfrac{1}{2})(\mu^*/k - \tfrac{1}{2}). \end{aligned} \tag{1.16}$$

Equating the right-hand side of (1.11) with (1.16), adding up the result over $\mu = a, M+a, 2M+a, \cdots, (k_1-1)M+a$, and using Lemma 1, we get

$$\begin{aligned} &\sum_{m=0}^{\infty} \{\lambda((mM+a)(z-ih)/k) + \lambda((mM+M-a)(z-ih)/k)\} \\ &\qquad + \pi m_1 z \sum_{\mu} \{(\mu/K)^2 - \mu/K + \tfrac{1}{6}\} \\ &= \sum_{m=0}^{\infty} \{\lambda((mD+b)(1/Kz - iH/k) + i\gamma a/M) \\ &\quad + \lambda((mD+D-b)(1/Kz - iH/k) - i\gamma a/M)\} \\ &\quad + (\pi/m_1 z) \sum_{\mu} \{(\mu^*/k)^2 - \mu^*/k + \tfrac{1}{6}\} + 2\pi i \sum_{\mu} (\mu/K - \tfrac{1}{2})(\mu^*/k - \tfrac{1}{2}). \end{aligned} \tag{1.17}$$

Here we have

$$\sum_{\mu} \{(\mu/K)^2 - \mu/K + \tfrac{1}{6}\} = (6a^2 - 6Ma + M^2)/6MK = A/6MK \tag{1.18}$$

by (1.7), and similarly

$$\sum_{\mu} \{(\mu^*/k)^2 - \mu^*/k + \tfrac{1}{6}\} = (6b^2 - 6Db + D^2)/6Dk = B/6Dk. \tag{1.19}$$

It is now obvious that the desired equation (1.8) follows from (1.17), (1.18) and (1.19).

This completes the proof of Theorem 1.

2. Estimation of some exponential sum. The transformation equation (1.8) contains the complicated root of unity $\omega(h,k)$ whose definition appears in (1.6).

We shall first express $\omega(h,k)$ in a more elementary form than (1.6), without making use of the function $[t]$.

Writing $\mu^* = h\mu - k[h\mu/k]$, we have from (1.6)

$$\begin{aligned} \sigma(h,k) &= \sum_{\mu} (\mu/K - \tfrac{1}{2})(\mu^*/k - \tfrac{1}{2}) \\ &= \sum_{\mu} (\mu/K)(\mu^*/k - \tfrac{1}{2}) - \tfrac{1}{2} \sum_{\mu} (\mu^*/k - \tfrac{1}{2}). \end{aligned} \tag{2.1}$$

Here, clearly

$$\begin{aligned}(2.2)\qquad \sum_{\mu}(\mu/K)(\mu^*/k-\tfrac{1}{2}) &= \sum_{\mu}(\mu/K)(h\mu/k-[h\mu/k]-\tfrac{1}{2})\\ &= (h/Kk)\sum_{\mu}\mu^2-(1/K)\sum_{\mu}\mu[h\mu/k]-(1/2K)\sum_{\mu}\mu,\end{aligned}$$

and further

$$\begin{aligned}(2.3)\qquad \sum_{\mu}\mu &= \sum_{l=0}^{k_1-1}(a+Ml) = ak_1+\tfrac{1}{2}Mk_1(k_1-1),\\ \sum_{\mu}\mu^2 &= \sum_{l=0}^{k_1-1}(a+Ml)^2 = a^2k_1+aMk_1(k_1-1)\\ &\qquad +\tfrac{1}{6}M^2k_1(k_1-1)(2k_1-1).\end{aligned}$$

Also, by Lemma 1,

$$\begin{aligned}(2.4)\qquad \sum_{\mu}(\mu^*/k-\tfrac{1}{2}) &= \sum_{l=0}^{k_1-1}((b+Dl)/k-\tfrac{1}{2})\\ &= bk_1/k+Dk_1(k_1-1)/2k-\tfrac{1}{2}k_1 = b/D-\tfrac{1}{2}.\end{aligned}$$

It follows from (2.1), (2.2), (2.3) and (2.4) that

$$\begin{aligned}(2.5)\qquad 12Mk\sigma(h,k) &= 2h\{6a^2+6aM(k_1-1)+M^2(k_1-1)(2k_1-1)\}\\ &\quad -3k\{2a+M(k_1-2)+2bm_1\}-12D\sum_{\mu}\mu[h\mu/k].\end{aligned}$$

Hence $12Mk\sigma(h,k)$ is always an integer. Moreover

$$(2.6)\qquad \sum_{\mu}\mu[h\mu/k] = \sum_{l=0}^{k_1-1}(a+Ml)[h\mu/k] = a\sum_{\mu}[h\mu/k]+M\sum_{l=0}^{k_1-1}l[h\mu/k] \qquad (\mu=a+Ml),$$

and

$$\begin{aligned}(2.7)\qquad \sum_{\mu}[h\mu/k] &= \sum_{\mu}(h\mu/k-\mu^*/k) = (h/k)\{ak_1+\tfrac{1}{2}Mk_1(k_1-1)\}\\ &\quad -(1/k)\{bk_1+\tfrac{1}{2}Dk_1(k_1-1)\}.\end{aligned}$$

From (2.5), (2.6) and (2.7) we obtain

$$\begin{aligned}12Mk\sigma(h,k) &= 2hM\{3a(k_1-1)+M(k_1-1)(2k_1-1)\}\\ &\quad -3K(k+2b-2D)+6a(2b-D)-12DM\sum_{l=0}^{k_1-1}l[h\mu/k].\end{aligned}$$

Multiplying both sides by $\gamma/12DM$ (γ being given in (1.1)), we deduce

$$\begin{aligned}(2.8)\qquad \gamma k_1\sigma(h,k) &\equiv (\gamma h/k)(aX+MY)-(\gamma k_1/4D)(k+2b-2D)\\ &\quad +(a\gamma/2DM)(2b-D) \pmod 1,\end{aligned}$$

where X, Y are integers defined by

$$(2.9)\qquad X=\tfrac{1}{2}k_1(k_1-1), \qquad Y=\tfrac{1}{6}k_1(k_1-1)(2k_1-1).$$

Now from (2.5) we see that

$$12Mk\sigma(h,k) \equiv 2hM^2(k_1-1)(2k_1-1) \pmod 3.$$

Here if $3 \nmid k$, then $3 \nmid k_1$ and so we have

$$12Mk\sigma(h,k) \equiv 0 \pmod 3 \text{ if } 3 \nmid k. \tag{2.10}$$

Next, from (2.5),

$$\begin{aligned} 12Mk\sigma(h,k) &\equiv 2hM^2(k_1-1)(2k_1-1) - 3k\{2a + M(k_1-2) + 2bm_1\} \\ &\equiv 2hM^2(k_1-1)(2k_1-1) + k(2a + m_1k + 2M + 2bm_1) \pmod 4. \end{aligned}$$

Here if $2 \nmid k$, then $2 \nmid k_1$ and therefore, noting that $k^2 \equiv 1 \pmod 4$ and that

$$k(2a + 2M + 2bm_1) \equiv 2a + 2M + 2bm_1 \pmod 4,$$

we have

$$12Mk\sigma(h,k) \equiv 2a + 2M + (2b+1)m_1 \pmod 4 \text{ if } 2 \nmid k. \tag{2.11}$$

We now introduce two integers f, g defined as follows:

$$\begin{aligned} &f = 12,\ g = 1 && \text{for } (k,6) = 1, \\ &f = 3,\ g = 4 && \text{for } (k,6) = 2, \\ &f = 4,\ g = 3 && \text{for } (k,6) = 3, \\ &f = 1,\ g = 12 && \text{for } (k,6) = 6. \end{aligned} \tag{2.12}$$

Then we see that $fg = 12$ and $(f,k) = 1$ in all cases. In addition, any prime in g divides k, so that $(h,k) = 1$ implies

$$(h, gDk) = 1, \tag{2.13}$$

and $(f,k) = 1$ implies

$$(f, gDk) = 1. \tag{2.14}$$

Furthermore, examination of (2.10), (2.11) and (2.12) yields

$$12Mk\sigma(h,k) \equiv 6a + 6M + 3(2b-1)m_1 \pmod f. \tag{2.15}$$

We next have, on the one hand,

$$\sum_{\mu} (\mu^*/k - \tfrac{1}{2})^2 = \sum_{\mu} (\mu^*/k)^2 - \sum_{\mu} (\mu^*/k) + \tfrac{1}{4}\sum_{\mu} 1$$

which reduces after simplification to

$$(B - k^2)/6Dk + k_1/4, \tag{2.16}$$

where B is given in (1.7); while, on the other hand,

$$\begin{aligned}\sum_{\mu}(\mu^*/k-\tfrac{1}{2})^2 &= \sum_{\mu}(h\mu/k-[h\mu/k]-\tfrac{1}{2})^2 = 2h\sum_{\mu}(\mu/k)(\mu^*/k-\tfrac{1}{2})\\ &\quad -(h/k)^2\sum_{\mu}\mu^2+\sum_{\mu}[h\mu/k]([h\mu/k]+1)+\tfrac{1}{4}\sum_{\mu}1\\ &= 2hm_1\sigma(h,k)+hm_1(b/D-\tfrac{1}{2})\\ &\quad -(h^2/6Dk)\{2K^2+3K(2a-M)+A\}+2S+k_1/4.\end{aligned} \tag{2.17}$$

where we have made use of (2.1), (2.4) and (2.3); A is given in (1.7), and S is an integer defined by

$$S=\tfrac{1}{2}\sum_{\mu}[h\mu/k]([h\mu/k]+1).$$

Comparing (2.16) and (2.17) yields

$$\begin{aligned}12Mhk\sigma(h,k) &= h^2\{2K^2+3K(2a-M)+A\}+(B-k^2)\\ &\quad +3hkm_1(D-2b)-12DkS.\end{aligned} \tag{2.18}$$

Let us now assume that H is any fixed solution of the congruence (cf. (1.2))

$$hH\equiv\delta \pmod{gDk}, \tag{2.19}$$

which is solvable by virtue of (2.13). Multiplying (2.18) by this H, and observing that $gDk \mid 12Dk$, we obtain

$$12Mk\delta\sigma(h,k)\equiv uh+vH+3\delta km_1(D-2b) \pmod{gDk}, \tag{2.20}$$

where

$$u=\delta\{2K^2+3K(2a-M)+A\},\qquad v=B-k^2. \tag{2.21}$$

We take here any integers ϕ, ψ satisfying

$$f\phi+gDk\psi=1, \tag{2.22}$$

which is possible in view of (2.14). Then, recalling that $fg=12$, we deduce from (2.15), (2.20) and (2.22) that

$$\begin{aligned}12Mk\delta\sigma(h,k) &\equiv f\phi\{uh+vH+3\delta km_1(D-2b)\}\\ &\quad +gDk\psi\delta\{6a+6M+3(2b-1)m_1\} \pmod{12Dk}.\end{aligned}$$

Dividing both sides by $12Dk$, we have

$$\begin{aligned}\delta m_1\sigma(h,k) &\equiv(\phi/gDk)(uh+vH)+(3\phi\delta m_1/gD)(D-2b)\\ &\quad +(g\psi\delta/4)\{2a+2M+(2b-1)m_1\} \pmod 1.\end{aligned} \tag{2.23}$$

It now follows from (2.8), (2.23) and (1.1) that

$$\begin{aligned}\sigma(h,k) &\equiv (1/gDk)(\{gD\gamma(aX+MY)-\phi u\}h-\phi vH)\\ &\quad -(\gamma k_1/4D)(k+2b-2D)+\{(a\gamma/2DM)+(3\phi\delta m_1/gD)\}(2b-D)\\ &\quad -(g\psi\delta/4)\{2a+2M+(2b-1)m_1\} \pmod 1\\ &= (1/gDk)(Uh+VH)+W,\end{aligned} \tag{2.24}$$

say. We remark here that U and V are integers, U being independent of h, and that if we keep b fixed and restrict h by the congruence $ah \equiv b \pmod D$ (see (1.5)), then V, W are also independent of such restricted values of h.

From (1.6) and (2.24) we obtain the desired expression:

$$\omega(h,k) = \exp\{(2\pi i/gDk)(Uh+VH)+2\pi iW\}. \tag{2.25}$$

We now proceed to consider the following exponential sum:

$$S_k = S_k(n,\nu;b,D;s_1,s_2) = \sum_h{}' \omega(h,k)\exp\{2\pi i(-nh+\nu H)/k\} \tag{2.26}$$
$$(h \bmod k, ah \equiv b \pmod D, s_1 \leqq (\bar h) < s_2).$$

Here n, ν, b, s_1, s_2 are all integers with $n \geqq 0$, $0 \leqq b < D$, $(b,D)=1$, $0 < s_2 - s_1 \leqq k$; H is defined by (1.2); $\bar h$ is any fixed solution of $h\bar h \equiv 1 \pmod k$; the notation $s_1 \leqq (\bar h) < s_2$ means that there exists an integer t such that $t \equiv \bar h \pmod k$, $s_1 \leqq t < s_2$; and the summation symbol $\sum'_h$ denotes here and subsequently that h runs over any reduced residue system of the given modulus with certain summation conditions indicated in parentheses.

First, we notice that each summand in S_k has a period k with respect to h, and hence the residue system modulo k over which the summation extends can be arbitrary.

Let us now define an arithmetic function $f(s)$ by

$$f(s) = \begin{cases} 1, & \text{if } s_1 \leqq (s) < s_2, \\ 0, & \text{otherwise.}\end{cases} \tag{2.27}$$

Then $f(s)$ is a periodic function with period k, so that it permits of expansion into a finite Fourier series of the form

$$f(s) = \sum_{l=0}^{k-1} c_l \exp(2\pi isl/k). \tag{2.28}$$

Here

$$c_l = k^{-1} \sum_{s \bmod k} f(s)\exp(-2\pi isl/k) = k^{-1} \sum_{s=s_1}^{s_2-1} \exp(-2\pi isl/k),$$

and so

$$c_0 = (s_2 - s_1)/k,$$

$$c_l = \csc(\pi l/k)\exp(\pi i l/k)\{\exp(-2\pi i s_1 l/k) - \exp(-2\pi i s_2 l/k)\}/2ik \qquad (l \neq 0)$$

hence we have

$$(2.29) \qquad \begin{aligned} &|c_0| \leqq 1, \\ &|c_l| \leqq \csc(\pi l/k)/k \leqq \begin{cases} 1/2l & (1 \leqq l \leqq k/2), \\ 1/2(k-l) & (k/2 \leqq l \leqq k-1). \end{cases} \end{aligned}$$

By (2.27), we may write

$$(2.30) \qquad S_k = \sum_h{}' f(\bar{h})\omega(h,k)\exp\{2\pi i(-nh + \nu H)/k\} \qquad (h \bmod k, ah \equiv b \pmod D).$$

Further, it is easy to see that the value of S_k is not affected by the selection of $\bar{h}$ and H as solutions of their congruences. Accordingly, we may employ those $\bar{h}$, H which satisfy

$$h\bar{h} \equiv 1 \pmod{gDk}, \qquad hH \equiv \delta \pmod{gDk}$$

(see (2.19)), respectively. Then we have

$$(2.31) \qquad H \equiv \delta\bar{h} \pmod{gDk}$$

and *a fortiori*

$$(2.32) \qquad H \equiv \delta\bar{h} \pmod k.$$

From (2.28), (2.30), (2.32) and the periodicity of the summands it follows that

$$\begin{aligned} S_k &= \sum_{l=0}^{k-1} c_l \sum_h{}' \omega(h,k)\exp(2\pi i\{-nh + (\nu\delta + l)\bar{h}\}/k) \qquad (h \bmod k, ah \equiv b \pmod D) \\ &= (gD)^{-1}\sum_{l=0}^{k-1} c_l \sum_h{}' \omega(h,k)\exp(2\pi i\{-gDnh + gD(\nu\delta + l)\bar{h}\}/gDk) \qquad (h \bmod gDk, ah \equiv b \pmod D). \end{aligned}$$

We define here an integer a' as any fixed solution of

$$(2.33) \qquad aa' \equiv 1 \pmod M.$$

Using (2.33), (2.31), (2.25) and the remark following (2.24), we get

$$(2.34) \qquad \begin{aligned} S_k = (gD)^{-1}\exp(2\pi i W)\sum_{l=0}^{k-1} c_l \sum_h{}' \exp(2\pi i\{&(U - gDn)h \\ &+ (V\delta + gD(\nu\delta + l))\bar{h}\}/gDk) \\ &(h \bmod gDk, h \equiv a'b \pmod D). \end{aligned}$$

The inner sum, $T_k(l)$ say, is a generalized Kloosterman sum, and therefore admits of the estimate (see [1], [11], [13]):

$$|T_k(l)| < C_0(gDk)^{1-\alpha}(gDk, U-gDn)^{\beta}, \tag{2.35}$$

where α, β are any numbers satisfying $0<\alpha<\beta<\frac{1}{2}$, and C_0 is a positive constant depending only on α and β.

Thus we obtain from (2.34), (2.35) and (2.29)

$$\begin{aligned} |S_k| &\leqq (gD)^{-1}\sum_{l=0}^{k-1}|c_l|\,|T_k(l)| \\ &< (gD)^{-1}C_0(gDk)^{1-\alpha}(gDk, U-gDn)^{\beta}\{1+2\sum_{l=1}^{\leqq k/2}(2l)^{-1}\} \\ &< C_0(gD)^{-\alpha}k^{1-\alpha}(gDk, U-gDn)^{\beta}\log(4k). \end{aligned} \tag{2.36}$$

On the other hand, by (2.24) and (2.21), we have

$$U = gD\gamma(aX+MY) - \phi\delta\{2K^2+3K(2a-M)+A\}.$$

But examination of (2.9) and (2.12) reveals that

$$\begin{aligned} gDX &= kg(k_1-1)/2 \equiv 0 \pmod{k}, \\ gDY &= kg(k_1-1)(2k_1-1)/6 \equiv 0 \pmod{k}. \end{aligned}$$

Hence

$$U \equiv -\phi\delta A \pmod{k}. \tag{2.37}$$

Using (2.37), (2.22), and recalling that $(f,k)=1$, $fg=12$, $(k_1,m_1)=1$ and (1.1), we find

$$\begin{aligned} (gDk, U-gDn) &\leqq gD(k, U-gDn) = gD(k, -\phi\delta A - gDn) \\ &= gD(k, f\phi\delta A + fgDn) = gD(k, \delta A + 12Dn) \\ &\leqq gD^2(k_1, \delta A + 12Dn) = gD^2(k_1, \delta m_1 A + 12m_1 Dn) \\ &= gD^2(k_1, (\gamma k_1 - 1)A + 12Mn) = gD^2(k_1, 12Mn - A) \\ &\leqq gD^2\,|12Mn - A|, \end{aligned} \tag{2.38}$$

where the last inequality follows from the fact that

$$12Mn - A \neq 0 \quad \text{for } n = 0, 1, 2, \cdots. \tag{2.39}$$

To prove (2.39), suppose on the contrary that $12Mn - A = 0$ for some value of n. Then $A \equiv 0 \pmod{M}$, i.e. $A = 6a^2 - 6Ma + M^2 \equiv 0 \pmod{M}$, so that $6a^2 \equiv 0 \pmod{M}$. But we have assumed that $(a, M) = 1$, and so $6 \equiv 0 \pmod{M}$, which implies $M \leqq 6$. Moreover, since $0 < a < M$, we have $A < M^2$. Hence $n = A/12M < M/12 \leqq 6/12 = \frac{1}{2}$. Therefore $n = 0$, and we have $A = 0$, which implies that a/M is an irrational number, and this is clearly a contradiction.

By using (2.38), it follows from (2.36) that[3]

$$|S_k| < C_0(gD)^{-\alpha}k^{1-\alpha}\{gD^2\,|\,12Mn-A\,|\}^{\beta}\log(4k)$$
$$= C_0 g^{\beta-\alpha}D^{2\beta-\alpha}k^{1-\alpha}\,|\,12Mn-A\,|^{\beta}\log(4k) = O(k^{1-\alpha}(n+1)^{\beta}\log k) \quad (n \geqq 0).$$

Here we can omit the $\log k$ from the O-term since we may choose α arbitrarily large so far as $\alpha < \beta$.

Thus we conclude the following

THEOREM 2. *The sum S_k defined by (2.26) is subject to the estimate*

$$|S_k| < Ck^{1-\alpha}(n+1)^{\beta} \quad (n \geqq 0),$$

where α, β are any numbers satisfying $0 < \alpha < \beta < \frac{1}{2}$, and C is a constant depending only on α, β and M.

3. A convergent series for $p(n; a, M)$. We are now ready to apply the Hardy-Ramanujan method. We shall treat the partition function $p(n; a, M)$ for $n \geqq 0$, provided that $p(0; a, M) = 1$.

In the first place, we obtain, by Cauchy's theorem,[4]

$$p(n; a, M) = \tfrac{1}{2\pi i}\int_C x^{-n-1}F(x; a, M)\,dx.$$

Using the Farey dissection of order N, this becomes

$$p(n; a, M) = \sum_{h,k}{}' \exp(-2\pi i n h/k) \cdot \int_{-\theta'}^{\theta''} F(\exp(2\pi i h/k - 2\pi w); a, M)\exp(2\pi n w)\,d\phi \tag{3.1}$$
$$(0 \leqq h < k \leqq N;\ \theta' = \theta'_{h,k},\ \theta'' = \theta''_{h,k};\ w = N^{-2} - i\phi).$$

Now, in the transformation equation (1.8) we expand $F(x'; b, D, \rho)$ (see (1.4)) into a power series in x' as

$$F(x'; b, D, \rho) = \prod_{m=0}^{\infty}(1-\rho x'^{mD+b})^{-1}(1-\rho^* x'^{mD+D-b})^{-1} = \sum_{\nu=0}^{\infty} c_\nu(b, D, \rho)x'^{\nu}, \tag{3.2}$$

and substitute the relation (1.3), putting $z = kw$. Then (1.8) becomes

$$F(\exp(2\pi i h/k - 2\pi w); a, M) = \omega(h, k)\exp\{(\pi/6Mk)(B/kw - Akw)\} \cdot \sum_{\nu=0}^{\infty} c_\nu(b, D, \rho)\exp(2\pi i \nu H/k - 2\pi\nu/Kkw).$$

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[3] We mean throughout the paper that the constant implied in the O-symbol always depends at most on α, β and M.

[4] For definitions of the unexplained notations in this section, see Rademacher [10].

Insertion of this into (3.1) yields

$$p(n;a,M)=\sum_{h,k}{}'\,\omega(h,k)\exp(-2\pi inh/k)\int_{-\theta'}^{\theta''}\sum_{\nu=0}^{\infty}c_\nu(b,D,\rho)$$
$$\cdot\exp\{(\pi/6Mk^2\mathrm{w})(B-12D\nu)-(\pi w/6M)(A-12Mn)\}\exp(2\pi i\nu H/k)d\phi. \tag{3.3}$$

We separate here the sum over ν into two parts as

$$\sum_{\nu=0}^{\infty}=\sum_{\nu=0}^{[B/12D]}+\sum_{\nu>[B/12D],\,\nu\geqq 0}$$

so that the coefficient of w^{-1} is always positive[5] in the first sum,[6] while it is negative in the second sum. The right member of (3.3) then splits into two parts according to the above separation. Let that one which corresponds to the first sum be $Q(n)$, and let the other be $R(n)$. Then it will be seen without difficulty, though we will not develop here the details, that, by using a method analogous to that of Rademacher [10] and applying the estimation in Theorem 2, one can obtain the following result:

$$p(n;a,M)=Q(n)+R(n),$$
$$Q(n)=2\pi\sum_{k=1}^{N}\sum_{\substack{0\leqq b<D\\(b,D)=1}}\sum_{\nu=0}^{[B/12D]}c_\nu(b,D,\rho)S_k\cdot L_k(n,\nu;B,D)$$
$$+O(N^{-\alpha}(n+1)^{\beta}\exp(2\pi nN^{-2})), \tag{3.4}$$
$$R(n)=O(N^{-\alpha}(n+1)^{\beta}\exp(2\pi nN^{-2})),$$

where

$$S_k=S_k(n,\nu;b,D;0,k)\quad(\text{see (2.26)}),$$

and[7]

$$L_k(n,\nu;B,D)$$
$$=\tfrac{1}{2\pi i}\int^{(0+)}\exp\{(\pi/6Mk^2w)(B-12D\nu)-(\pi w/6M)(A-12Mn)\}dw$$
$$=\begin{cases}k^{-1}(A-12Mn)^{-\frac{1}{2}}(B-12D\nu)^{\frac{1}{2}}J_1((\pi/3Mk)(A-12Mn)^{\frac{1}{2}}(B-12D\nu)^{\frac{1}{2}})\\ \qquad\qquad\text{if } n<A/12M,\\ k^{-1}(12Mn-A)^{-\frac{1}{2}}(B-12D\nu)^{\frac{1}{2}}I_1((\pi/3Mk)(12Mn-A)^{\frac{1}{2}}(B-12D\nu)^{\frac{1}{2}})\\ \qquad\qquad\text{if } n>A/12M,\end{cases} \tag{3.5}$$

$J_1(t)$, $I_1(t)$ being the Bessel functions of the first order.

[5] The fact that $B-12D\nu\neq 0$ for $\nu=0,1,2,\cdots$ may be verified in the same way as (2.39).

[6] If $B<0$, this sum becomes empty and is to be interpreted as zero.

[7] See Watson [12], p. 176, (1) and p. 181, (1). Notice that $J_1(z)=-J_{-1}(z)$ and $I_1(z)=I_{-1}(z)$.

In (3.4) we now divide the sum over k into two parts according as $D>1$ or $D=1$, noting that $D>1$ implies $b>0$, and that $D=1$ implies $b=0$, $B=1$. We then obtain from (3.4), by letting $N\to\infty$ for every fixed $n\geqq 0$,

$$\begin{aligned} p(n;a,M) &= 2\pi \sum_k^{(1)} \sum_{\substack{1\leqq b<D \\ (b,D)=1}} \sum_{\nu=0}^{[B/12D]} c_\nu(b,D,\rho) S_k \cdot L_k(n,\nu;B,D) \\ &\quad + 2\pi \sum_k^{(2)} c_0(0,1,\rho) S_k' \cdot L_k(n,0;1,1) \\ &= p^{(1)}(n;a,M) + p^{(2)}(n;a,M), \end{aligned} \tag{3.6}$$

say,[8] where the sums $\sum^{(1)}{}_k$ and $\sum^{(2)}{}_k$ are taken over those k for which $D=(k,M)>1$ and $D=1$ respectively, and where

$$S_k' = S_k(n,0;0,1;0,k). \tag{3.7}$$

Now let us consider $p^{(1)}(n;a,M)$ first. We see that $B/12D\geqq 0$ is equivalent to $|\frac{1}{2}D-b|\geqq 3^{\frac{1}{2}}D/6$, since

$$B = 6b^2 - 6Db + D^2 = 6(\tfrac{1}{2}D-b)^2 - \tfrac{1}{2}D^2$$

by (1.7). Therefore we can write

$$\begin{aligned} p^{(1)}(n;a,M) &= 2\pi \sum_k^{(1)} \sum_{\substack{b=1 \\ (b,D)=1}}^{[\lambda D]} \sum_{\nu=0}^{[B/12D]} \{c_\nu(b,D,\rho) S_k(n,\nu;b,D;0,k) \\ &\quad + c_\nu(D-b,D,\rho) S_k(n,\nu;D-b,D;0,k)\} L_k(n,\nu;B,D) \end{aligned} \tag{3.8}$$

with the abbreviation $\lambda=(3-3^{\frac{1}{2}})/6$, where we have made use of the fact that B is unchanged when b is replaced by $D-b$. Further, from (3.2) we have, since $0\leqq\nu<B/12D<D/12<D$,

$$c_\nu(b,D,\rho) = \sum_{l,l'} \rho^l \rho^{*l'},$$

where the summation extends over all non-negative integers l, l' satisfying

$$lb + l'(D-b) = \nu. \tag{3.9}$$

But, since $0\leqq\nu<\frac{1}{2}D$, $0<b<\frac{1}{2}D$, the only possible solution of (3.9) is $l'=0$, $l=\nu/b$ with $b\mid\nu$. Hence

$$c_\nu(b,D,\rho) = \begin{cases} \rho^{\nu/b}, & \text{if } b\mid\nu, \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

Similarly

$$c_\nu(D-b,D,\rho) = \begin{cases} \rho^{*\nu/b}, & \text{if } b\mid\nu, \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

[8] The series for $p^{(1)}(n;a,M)$ and $p^{(2)}(n;a,M)$ are in fact *absolutely* convergent, as may be seen from arguments in Section 4.

Hence, putting $\nu = br$, where $r = 0, 1, 2, \cdots, [B/12Db]$, we get from (3.8), (3.10) and (3.11)

$$p^{(1)}(n; a, M) = 2\pi \sum_{k}^{(1)} \sum_{\substack{b=1 \\ (b, D)=1}}^{[\lambda D]} \sum_{r=0}^{[B/12Db]} A_k(n, r; b, D) L_k(n, br; B, D), \tag{3.12}$$

where

$$A_k(n, r; b, D) = \sum_{\substack{h \bmod k \\ h \equiv \pm a' b (D)}} \rho^{\pm r} \omega(h, k) \exp\{2\pi i(-nh + brH)/k\} \tag{3.13}$$

by (2.26) and (2.33).

We remark further that the sum over b on the right of (3.12) is empty when $D = 2, 3, 4$. For, since $\lambda D = (3 - 3^{\frac{1}{2}}) D/6 = 1.26 \cdots \times D/6$, we have $[\lambda D] = 0$ for $D = 2, 3, 4$, and $[\lambda D] \geqq 1$ for $D \geqq 5$.

We shall next discuss $p^{(2)}(n; a, M)$. By (2.26), the sum S_k' of (3.7) contains the root of unity $\omega(h, k)$ whose definition is, by (1.6),

$$\omega(h, k) = \exp\{2\pi i \sigma(h, k)\} \tag{3.14}$$

with

$$\sigma(h, k) = \sum_{\mu} (\mu/Mk - \tfrac{1}{2})(h\mu/k - [h\mu/k] - \tfrac{1}{2}) \tag{3.15}$$
$$(\mu = a, M + a, 2M + a, \cdots, (k-1)M + a),$$

since $D = 1$ implies $K = Mk$, $k_1 = k$.

Now let ξ be an integer defined by

$$\xi k \equiv a \pmod{M} \quad (0 < \xi < M), \tag{3.16}$$

which is uniquely solvable since $D = (k, M) = 1$. Let $\mu_0 = \xi k$. Then we see that $h\mu/k$ is integer if and only if $\mu = \mu_0$ because of $(h, k) = 1$, $\mu \equiv a \pmod{M}$ and $0 < \mu < Mk$.

Consequently we can write (3.15) in the form

$$\begin{aligned} \sigma(h, k) &= \sum_{\mu \neq \mu_0} (\mu/Mk - \tfrac{1}{2})(h\mu/k - [h\mu/k] - \tfrac{1}{2}) + (\xi/M - \tfrac{1}{2})(-\tfrac{1}{2}) \\ &= \tau(h, k) + \tfrac{1}{2}(\tfrac{1}{2} - \xi/M), \end{aligned} \tag{3.17}$$

where

$$\tau(h, k) = \sum_{\mu} ((\mu/Mk))((h\mu/k)) \tag{3.18}$$
$$(\mu = a, M + a, 2M + a, \cdots, (k-1)M + a)$$

with the notation

$$((t)) = \begin{cases} 0, & \text{if } t \text{ is an integer,} \\ t - [t] - \tfrac{1}{2}, & \text{otherwise.} \end{cases} \tag{3.19}$$

It therefore follows from (3.14) and (3.17) that

$$\omega(h, k) = i \exp(-\pi i \xi/M) \chi(h, k) \tag{3.20}$$

with

$$\chi(h,k) = \exp\{2\pi i\tau(h,k)\} \tag{3.21}$$

On the other hand, (1.1) becomes

$$\gamma k - \delta M = 1, \tag{3.22}$$

since $D=1$ implies $k_1 = k$, $m_1 = M$. From (3.16) and (3.22) it follows that $\gamma a \equiv \xi \pmod{M}$, and hence, by the definition of ρ in (1.5), we have $\rho = \exp(-2\pi i\xi/M)$, so that

$$(1-\rho)^{-1} = (2i)^{-1}\csc(\pi\xi/M)\exp(\pi i\xi/M). \tag{3.23}$$

Further, it is evident from (3.2) that

$$c_0(0,1,\rho) = (1-\rho)^{-1}. \tag{3.24}$$

Thus we infer from (3.24), (3.23), (3.7), (2.26) and (3.20) that

$$c_0(0,1,\rho)S_k' = \tfrac{1}{2}\csc(\pi\xi/M)B_k(n), \tag{3.25}$$

where

$$B_k(n) = \sum_{h \bmod k}{}' \chi(h,k)\exp(-2\pi i n h/k) \tag{3.26}$$

with the $\chi(h,k)$ of (3.21). From (3.6) and (3.25) we obtain the expression:

$$p^{(2)}(n;a,M) = \pi \sum_{k}{}^{(2)} \csc(\pi\xi/M)B_k(n)L_k(n,0;1,1). \tag{3.27}$$

Finally, we define an integer k' as any fixed solution of

$$kk' \equiv 1 \pmod{M}, \tag{3.28}$$

which, combined with (3.16), yields that $\xi \equiv ak' \pmod{M}$. Hence

$$\csc(\pi\xi/M) = |\csc(\pi ak'/M)|. \tag{3.29}$$

We note, in addition, that when $D>1$ we can express $\sigma(h,k)$ in a form corresponding to (3.18), namely,

$$\sigma(h,k) = \sum_{\mu}((\mu/K))((h\mu/k)) \tag{3.30}$$
$$(\mu = a, M+a, 2M+a, \cdots, (k_1-1)M+a).$$

For we see that $0<\mu/K<1$ and that $D>1$ implies $h\mu/k \neq$ integer since the congruence $\xi k \equiv a \pmod{M}$ has no solution in ξ as we have assumed that $(a,M)=1$, and the result follows from (1.6) and (3.19).

On combining (3.6), (3.12), (3.27), (3.29) and the remark following (3.13) we obtain our main result (cf. [6], Theorem 4):

THEOREM 3. *If $p(n;a,M)$ denotes, when $n \geqq 1$, the number of partitions of n into positive summands congruent to $\pm a \pmod M$, and $p(0;a,M) = 1$, where $M \geqq 3$, $0 < a < M$, $(a,M) = 1$, then we have, for $n \geqq 0$, the convergent series representation*

$$p(n;a,M) = 2\pi \sum_{\substack{k>0 \\ D \geqq 5}} \sum_{\substack{b=1 \\ (b,D)=1}}^{[\lambda D]} \sum_{r=0}^{[B/12Db]} A_k(n,r;b,D) L_k(n,br;B,D) + \pi \sum_{\substack{k>0 \\ D=1}} |\csc(\pi ak'/M)|\, B_k(n) L_k(n,0;1,1),$$

where $A_k(n,r;b,D)$ and $B_k(n)$ are given by (3.13) *and* (3.26) *with $\omega(h,k)$ and $\chi(h,k)$ defined by* (1.6), (3.30) *and* (3.21), (3.18) *respectively; $L_k(n,br;B,D)$ and $L_k(n,0;1,1)$ by* (3.5); *k' by* (3.28); *and $\lambda = (3-3^{\frac{1}{2}})/6$.*

Remark. The sums A_k, B_k defined by (3.13), (3.26) have the following properties:

(i) A_k, B_k are always *real.*

(ii) The value of A_k is *independent* of the choice of γ, δ in (1.1).

(i) may easily be verified, e.g. for A_k, if in (3.13) one divides the sum into two parts according as $h \equiv +a'b \pmod D$ or $h \equiv -a'b \pmod D$, observing that the divided two sums are conjugate complex numbers because the $\sigma(h,k)$ of (3.30) and the H of (1.2) are both odd functions in the variable h (H is to be considered modulo k). Similarly for B_k.

Referring to (ii) we note that, although the formula (3.13) involves ρ and H which, by (1.5) and (1.2), depend upon the choice of γ, δ, the value of A_k remains unaffected by ρ, H. This follows from the fact that in (3.13) $\rho^{\pm r}\exp(2\pi ibrH/k)$ are actually independent of ρ and H. To show this, we make use of (1.15). Now let that value of μ for which $\mu^* = b$ be μ_b. Then (1.15) gives

$$\mu_b/K \equiv -Hb/k + \gamma a/M \pmod 1,$$

so that

$$\exp(-2\pi i\mu_b/K) = \rho \exp(2\pi ibH/k) \tag{3.31}$$

by the definition of ρ in (1.5). Next, let that value of μ for which $\mu^* = (k_1 - 1)D + b = k - (D-b)$ be $\mu_{k-(D-b)}$. Then, by (1.15),

$$\mu_{k-(D-b)}/K \equiv H(D-b-k)/k + \gamma a/M \pmod 1,$$

so that

$$\exp(2\pi i\mu_{k-(D-b)}/K) = \rho^{-1}\exp(2\pi i(D-b)H/k),$$

or, replacing $D-b$ by b,

$$\exp(2\pi i\mu_{k-b}/K)=\rho^{-1}\exp(2\pi ibH/k). \tag{3.32}$$

The desired result then follows from (3.31) and (3.32) on taking the r-th powers.

4. Asymptotic properties of $p(n;a,M)$. We require first a lemma on $I_1(t)$, the Bessel function of the first order with purely imaginary argument.

LEMMA 4.1.

$$I_1(t) \text{ is positive and increasing for } t>0. \tag{4.1}$$

$$I_1(t)=O(t) \quad (t\to 0). \tag{4.2}$$

$$I_1(t)=(2\pi t)^{-\frac{1}{2}}e^t(1+O(t^{-1})) \quad (t\to+\infty). \tag{4.3}$$

Proof. (4.1) and (4.2) are easy consequences of the expansion:

$$I_1(t)=\sum_{m=0}^{\infty}(\tfrac{1}{2}t)^{2m+1}/m!(m+1)!$$

(see Watson [12], p. 77, (2));

and (4.3) is obtained from [12], p. 203.

The following lemma will be also needed.

LEMMA 4.2. *Let*

$$S_1(N)=\sum_{\nu=1}^{N}\nu^{\frac{1}{2}}e^{N/\nu}, \qquad S_2(N)=\sum_{\nu=2}^{N}\nu^{\frac{1}{2}}e^{N/\nu}.$$

Then $S_1(N)<2e^N$ *for* $N\geqq 6$, *and* $S_2(N)<5e^{\frac{1}{2}N}$ *for* $N\geqq 12$.

Proof. Since $t^{\frac{1}{2}}e^{N/t}$ is decreasing for $1\leqq t\leqq N$, we have

$$S_1(N)\leqq e^N+\int_1^N t^{\frac{1}{2}}e^{N/t}\,dt=e^N+N^{\frac{3}{2}}\int_1^N y^{-\frac{5}{2}}e^y\,dy \quad (N/t=y). \tag{4.4}$$

Here, since $y^{-\frac{5}{2}}e^y$ is decreasing for $1\leqq y\leqq\frac{5}{2}$ and increasing for $y\geqq\frac{5}{2}$, it follows that

$$y^{-\frac{5}{2}}e^y\leqq\max(e,N^{-\frac{5}{2}}e^N) \quad (1\leqq y\leqq N). \tag{4.5}$$

But we have $-\frac{5}{2}\log 6+6=1.5\cdots>1$, so that $N^{-\frac{5}{2}}e^N>e$ for $N\geqq 6$. Hence, by (4.5),

$$y^{-\frac{5}{2}}e^y\leqq N^{-\frac{5}{2}}e^N \quad (1\leqq y\leqq N, N\geqq 6).$$

Therefore

$$\int_1^N y^{-\frac{1}{2}} e^y \, dy < N \cdot N^{-\frac{1}{2}} e^N = N^{-\frac{1}{2}} e^N \quad (N \geqq 6),$$

which, together with (4.4), yields the first inequality in our lemma. The second may be proved similarly.

We now determine the asymptotic behavior of $p(n; a, M)$ as n becomes large. By Theorem 3, we have

$$p(n; a, M) = p^{(1)}(n; a, M) + p^{(2)}(n; a, M), \tag{4.6}$$

where

$$p^{(1)}(n; a, M) = 2\pi \sum_k^{(1)} \sum_{\substack{b=1 \\ (b, D)=1}}^{[\lambda D]} \sum_{r=0}^{[B/12Db]} A_k(n, r; b, D) L_k(n, br; B, D), \tag{4.7}$$

$$p^{(2)}(n; a, M) = \pi \sum_k^{(2)} |\csc(\pi a k'/M)| B_k(n) L_k(n, 0; 1, 1), \tag{4.8}$$

$$\left(\sum_k^{(1)} = \sum_{k>0,\, D\geqq 5}, \sum_k^{(2)} = \sum_{k>0,\, D=1} \right)$$

in which

$$L_k(n, br; B, D) = (kE)^{-1}(B - 12Dbr)^{\frac{1}{2}} I_1\left((\pi E/3Mk)(B - 12Dbr)^{\frac{1}{2}}\right) \tag{4.9}$$

$$L_k(n, 0; 1, 1) = (kE)^{-1} I_1(\pi E/3Mk) \tag{4.10}$$

with the abbreviation

$$E = (12Mn - A)^{\frac{1}{2}} \quad (n > A/12M). \tag{4.11}$$

Let us begin with $p^{(1)}(n; a, M)$. Clearly it suffices to consider the case $M \geqq 5$. Now if we keep D fixed, B takes its maximum value when $b = 1$, i.e., the value $B_1 = 6 - 6D + D^2$, since

$$B = 6b^2 - 6Db + D^2 = 6(\tfrac{1}{2}D - b)^2 - \tfrac{1}{2}D^2$$

and $1 \leqq b < \frac{1}{2}D$. Hence from (4.9) and (4.1) we get

$$L_k(n, br; B, D) \leqq (kE)^{-1}(B - 12Dbr)^{\frac{1}{2}} I_1(\pi E B_1^{\frac{1}{2}}/3Mk). \tag{4.12}$$

On the other hand, from (3.13) we have trivially

$$|A_k(n, r; b, D)| \leqq \sum_{\substack{h \bmod k \\ h \equiv \pm a' b (D)}}' 1 < k, \tag{4.13}$$

while, by Theorem 2, we have the non-trivial estimate [9]

$$|A_k(n, r; b, D)| < C_1 k^{1-\alpha}(n+1)^{\beta} \quad (0 < \alpha < \beta < \tfrac{1}{2}). \tag{4.14}$$

[9] In what follows we shall use $C_1, C_2, \cdots, C_{10}$ to denote certain positive constants depending at most on α, β and M.

From (4.12), (4.13) and (4.14) it follows that

$$|\sum_{\substack{b=1\\(b,D)=1}}^{[\lambda D]}\sum_{r=0}^{[B/12Db]} A_k(n,r;b,D)L_k(n,br;B,D)| < \begin{cases} E^{-1}Z_D I_1(\pi EB_1^{\frac{1}{3}}/3Mk), \\ C_1E^{-1}k^{-\alpha}(n+1)^{\beta}Z_D I_1(\pi EB_1^{\frac{1}{3}}/3Mk) \end{cases} \tag{4.15}$$

with the abbreviation

$$Z_D = \sum_{\substack{b=1\\(b,D)=1}}^{[\lambda D]}\sum_{r=0}^{[B/12Db]} (B-12Dbr)^{\frac{1}{3}}.$$

Using (4.15) and noting that

$$\sum_k^{(1)} = \sum_{\substack{D\mid M\\D\geqq 5}}\sum_{\substack{k_1=1\\(k_1,m_1)=1}}^{\infty},$$

where $k=k_1D$, $M=m_1D$, we get, by (4.7),

$$\begin{aligned} |p^{(1)}(n;a,M)| &< 2\pi\sum_{\substack{D\mid M\\D\geqq 5}}\sum_{k_1=1}^{\leqq F} E^{-1}Z_D I_1(F/k_1) \\ &\quad + 2\pi\sum_{\substack{D\mid M\\D\geqq 5}}\sum_{k_1>F} C_1E^{-1}D^{-\alpha}k_1^{-\alpha}(n+1)^{\beta}Z_D I_1(F/k_1), \end{aligned} \tag{4.16}$$

where

$$F=\pi EB_1^{\frac{1}{3}}/3MD. \tag{4.17}$$

But it follows from (4.2), (4.3) and Lemma 4.2 that

$$\begin{aligned} \sum_{k_1=1}^{\leqq F} I_1(F/k_1) &< C_2\sum_{k_1=1}^{\leqq F}(2\pi F/k_1)^{-\frac{1}{2}}e^{F/k_1} \\ &= C_2(2\pi F)^{-\frac{1}{2}}\sum_{k_1=1}^{\leqq F} k_1^{\frac{1}{2}}e^{F/k_1} < C_3F^{-\frac{1}{2}}e^F, \end{aligned} \tag{4.18}$$

$$\sum_{k_1>F} k_1^{-\alpha}I_1(F/k_1) < C_4F\sum_{k_1>F} k_1^{-1-\alpha} < C_5F^{1-\alpha}.$$

Now, set

$$G=\pi E/3M. \tag{4.19}$$

Then we have

$$F=GB_1^{\frac{1}{3}}/D \tag{4.20}$$

by (4.17). Further, since $5\leqq D\leqq M$ and

$$B_1/D^2 = 1-6/D+6/D^2 = 6(\tfrac{1}{2}-1/D)^2-\tfrac{1}{2},$$

we have

$$6(\tfrac{1}{2}-\tfrac{1}{5})^2-\tfrac{1}{2}\leqq B_1/D^2\leqq 6(\tfrac{1}{2}-1/M)^2-\tfrac{1}{2},$$

i.e.,

$$\tfrac{1}{25} \leqq B_1/D^2 \leqq M_1^2,$$

where

$$M_1 = (1 - 6/M + 6/M^2)^{\frac{1}{3}} < 1. \tag{4.21}$$

Thus we obtain $\frac{1}{5} \leqq B_1^{\frac{1}{3}}/D \leqq M_1 < 1$, and hence, noting (4.20),

$$G/5 \leqq F \leqq M_1 G < G. \tag{4.22}$$

From (4.16), (4.18) and (4.22) we deduce

$$\begin{aligned} |p^{(1)}(n;a,M)| &< 2\pi C_3 E^{-1}(G/5)^{-\frac{1}{3}} e^{M_1 G} \sum_{\substack{D \mid M \\ D \geqq 5}} Z_D \\ &\quad + 2\pi C_1 C_5 E^{-1} G^{1-\alpha}(n+1)^{\beta} \sum_{\substack{D \mid M \\ D \geqq 5}} D^{-\alpha} Z_D \\ &= C_6 E^{-1} G^{-\frac{1}{3}} e^{M_1 G} + C_7 E^{-1} G^{1-\alpha}(n+1)^{\beta}. \end{aligned} \tag{4.23}$$

We next treat $p^{(2)}(n;a,M)$. In (4.8) we distinguish the term $k=1$ from the others, and obtain

$$\begin{aligned} p^{(2)}(n;a,M) &= \pi \csc(\pi a/M) B_1(n) L_1(n,0;1,1) \\ &\quad + \pi \sum_{k>1}^{(2)} |\csc(\pi a k'/M)| B_k(n) L_k(n,0;1,1). \end{aligned} \tag{4.24}$$

Here evidently

$$|\csc(\pi a k'/M)| \leqq \csc(\pi/M) < \tfrac{1}{2}\pi(M/\pi) = \tfrac{1}{2}M. \tag{4.25}$$

Further we get

$$|B_k(n)| < k, \tag{4.26}$$

$$|B_k(n)| < C_8 k^{1-\alpha}(n+1)^{\beta} \qquad (0 < \alpha < \beta < \tfrac{1}{2}), \tag{4.27}$$

which correspond to (4.13), (4.14).

From (4.10), (4.19), (4.25), (4.26), (4.27) we obtain

$$\begin{aligned} &|\pi \sum_{k>1}^{(2)} |\csc(\pi a k'/M)| \, B_k(n) L_k(n,0;1,1)| \\ &\quad < \pi \sum_{k=2}^{\leqq G} \tfrac{1}{2} M k (kE)^{-1} I_1(G/k) \\ &\qquad + \pi \sum_{k>G} \tfrac{1}{2} M C_8 k^{1-\alpha}(n+1)^{\beta} I_1(G/k) \\ &\quad < C_9 E^{-1} G^{-\frac{1}{3}} e^{\frac{1}{3}G} + C_{10} E^{-1} G^{1-\alpha}(n+1)^{\beta}, \end{aligned} \tag{4.28}$$

where in the last step we have made use of (4.2), (4.3) and Lemma 4.2.

On the other hand, we have

$$B_1(n) = 1 \tag{4.29}$$

by (3.26), and

$$L_1(n,0;1,1) = E^{-1}I_1(G) = E^{-1}(2\pi G)^{-\frac{1}{2}}e^G(1+O(G^{-1})) \tag{4.30}$$

by (4.3).

It now follows from (4.6), (4.23), (4.24), (4.28), (4.29) and (4.30) that

$$p(n;a,M) = \pi\csc(\pi a/M)E^{-1}I_1(G) \cdot \{1+O(e^{M_1G-G})+O(e^{-\frac{1}{2}G})+O(G^{\frac{3}{2}-\alpha}(n+1)^{\beta}e^{-G})\}$$

Substituting from (4.11), (4.19), we are led to the following

THEOREM 4. *We have*

$$p(n;a,M) = \pi\csc(\pi a/M)\cdot(12Mn-A)^{-\frac{1}{4}} \cdot I_1((\pi/3M)(12Mn-A)^{\frac{1}{2}})\{1+O(\exp(-cn^{\frac{1}{2}}))\},$$

where c is a positive constant defined by

$$c = 2\pi(3M)^{-\frac{1}{2}}\min(\tfrac{1}{2}, 1-M_1)$$

with the M_1 of (4.21).

From Theorem 4 and (4.3), we can easily find an asymptotic formula for $p(n;a,M)$ in terms of elementary functions of n. We have

$$p(n;a,M) = \csc(\pi a/M)\cdot(3M/2)^{\frac{1}{2}}(12Mn-A)^{-\frac{3}{4}} \cdot \exp((\pi/3M)(12Mn-A)^{\frac{1}{2}})\{1+O(n^{-\frac{1}{2}})\},$$

or further

$$p(n;a,M) = \tfrac{1}{4}\csc(\pi a/M)\cdot(3M)^{-\frac{1}{4}}n^{-\frac{3}{4}}\exp(2\pi(n/3M)^{\frac{1}{2}})\{1+O(n^{-\frac{1}{2}})\}. \tag{4.31}$$

This formula will also be deduced, apart from the error term, from a result of Meinardus [7].

Finally, it is readily seen from (4.31) that, for any two values a_1, a_2 of a, we have

$$\lim_{n\to\infty}(p(n;a_1,M) : p(n;a_2,M)) = \csc(\pi a_1/M) : \csc(\pi a_2/M).$$

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DIFFERENTIABLE STRUCTURES ON SPHERES.*

By JOHN MILNOR.[1]

According to [5] the sphere S^7 can be given several differentiable structures which are essentially distinct. A corresponding result for the 15-sphere has been proved by Shimada [10] and Tamura [12]. The object of this note is to prove corresponding theorems for other dimensions of the form $4k-1$.

In § 1 certain differentiable manifolds $M(f_1, f_2)$ are constructed and studied; where $(f_1) \in \pi_m(SO_{n+1})$, $(f_2) \in \pi_n(SO_{m+1})$. In most cases these manifolds are topologically spheres. In § 2 an invariant λ is defined for differentiable $(4k-1)$-manifolds which are both homology spheres and boundaries. In § 3 the invariant $\lambda(M(f_1, f_2))$ is computed.

For $k \leqq 8$ the calculations are carried out explicitly. It is shown that there exist non-standard differentiable structures on S^{4k-1} for $k = 2, 4, 5, 6, 7, 8$. For example S^{31} has over sixteen million distinct differentiable structures. It is conjectured that the same argument works for all $k \geqq 4$; but I have not succeeded in solving the number theoretic problem which arises.

For $k = 1, 3$ the argument does not work. This is not surprising in the case $k = 1$, since J. Munkres, S. Smale, and J. H. C. Whitehead have shown (independently) that two differentiable 3-manifolds which are homeomorphic must necessarily be diffeomorphic.

The word *manifold* will always be used for a compact, oriented manifold, with or without boundary. The symbol D^k will stand for the unit disk in the euclidean space R^k.

1. Construction of manifolds homeomorphic to spheres. Given any diffeomorphism [2] $f: S^m \times S^n \to S^m \times S^n$, a manifold M of dimension $m+n+1$ is obtained by matching the boundaries of $D^{m+1} \times S^n$ and $S^m \times D^{n+1}$ under the correspondence f. That is: M is the identification space obtained from the disjoint union of $D^{m+1} \times S^n$ and $S^m \times D^{n+1}$ by identifying each point (x, y) in the boundary of $D^{m+1} \times S^n$ with $f(x, y)$, considered as a point in boundary of $S^m \times D^{n+1}$.

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[1] The author holds a Sloan fellowship.

[2] A *diffeomorphism* is a differentiable homeomorphism with differentiable inverse.

An alternative definition of M, which makes it into a differentiable manifold, is the following. Let $f(x,y)=(x',y')$ and define $t'=1/t$. Start with disjoint spaces $R^{m+1}\times S^n$ and $S^m\times R^{n+1}$. Let M be obtained from these by matching (tx,y) with $(x',t'y')$ for every $x\in S^m$, $y\in S^n$, $0<t<\infty$.

[As an example suppose that f is the identity map of $S^m\times S^n$. Then M is diffeomorphic to the unit sphere $S^{m+n+1}\subset R^{m+1}\times R^{n+1}$. In fact the correspondence

$$(tx,y)\to(tx/(1+t^2)^{\frac{1}{2}},y/(1+t^2)^{\frac{1}{2}}),$$
$$(x',t'y')\to(x'/(1+t'^2)^{\frac{1}{2}},t'y'/(1+t'^2)^{\frac{1}{2}})$$

defines a diffeomorphism $M\to S^{m+n+1}$.]

If $y\in S^n$ has coordinates $(y_0,\cdots,y_n)$, define $h(y)=y_n$. The function $h\colon S^n\to[-1,1]$ has just two critical points.

LEMMA 1. *Suppose that the diffeomorphism*

$$(x,y)\xrightarrow{f}(x',y')$$

satisfies the restriction $h(y)=h(y')$ for all (x,y). Then the manifold M constructed above is homeomorphic to S^{m+n+1}.

Proof. A differentiable map $g\colon M\to[-1,1]$ is defined by the correspondence

$$(tx,y)\to h(y)/(1+t^2)^{\frac{1}{2}}\ \text{(in the first coordinate system)},$$
$$(x',t'y')\to t'h(y')/(1+t'^2)^{\frac{1}{2}}\ \text{(in the second)}.$$

It is easily verified that g has just two critical points, and that these are non-degenerate. Together with [5] Theorem 2, this completes the proof.

One way to construct such a diffeomorphism $(x,y)\to(x',y')$ is the following. Start with differentiable maps of spheres into rotation groups

$$f_1\colon S^m\to SO_{n+1},\qquad f_2\colon S^n\to SO_{m+1},$$

and let

$$y'=f_1(x)\cdot y,\qquad x'=f_2(y')^{-1}\cdot x=f_2(f_1(x)\cdot y)^{-1}\cdot x$$

for all $x\in S^m$, $y\in S^n$. This defines a diffeomorphism with inverse

$$x=f_2(y')\cdot x',\qquad y=f_1(x)^{-1}\cdot y'=f_1(f_2(y')\cdot x')^{-1}\cdot y'.$$

[More generally the rotation groups SO_{n+1} and SO_{m+1} could be replaced by the groups Diff S^n, Diff S^m consisting of all diffeomorphisms.] The condition $h(y)=h(y')$ is equivalent to the requirement that $f_1(S^m)$ be contained in

the subgroup $SO_n \subset SO_{n+1}$. Whether this condition is satisfied or not, *the resulting* $(m+n+1)$*-manifold will be denoted by* $M(f_1, f_2)$.

Next we will show that $M(f_1, f_2)$ is a boundary. Start with three copies of the space $D^{m+1} \times D^{n+1}$. The notation (x_i, y_i) will be used for a point of $(D^{m+1} \times D^{n+1})_i$, $i = 1, 2, 3$. Identify $(S^m \times D^{n+1})_1$ with $(S^m \times D^{n+1})_2$ by the correspondence $(x_1, y_1) \to (x_2, y_2)$ where

$$x_2 = x_1, \qquad y_2 = f_1(x_1) \cdot y_1.$$

The resulting space $(D^{m+1} \times D^{n+1})_1 \cup (D^{m+1} \times D^{n+1})_2$ can be considered as a fibre bundle over the $(m+1)$-sphere $(D^{m+1})_1 \cup (D^{m+1})_2$ with fibre D^{n+1}, group SO_{n+1}, and characteristic map f_1. (See Steenrod [11] § 18.) It follows that this union can be given a differentiable structure in a natural way.

Next identify $(D^{m+1} \times S^n)_2$ with $(D^{m+1} \times S^n)_3$ by the correspondence $(x_2, y_2) \leftarrow (x_3, y_3)$, where $y_2 = y_3$, $x_2 = f_2(y_3) \cdot x_3$. Thus

$$(D^{m+1} \times D^{n+1})_2 \cup (D^{m+1} \times D^{n+1})_3$$

becomes a fibre bundle over an $(n+1)$-sphere with fibre D^{m+1} and characteristic map f_2.

Let W_1 denote the union of all three copies of $D^{m+1} \times D^{n+1}$. Clearly W_1 is a topological manifold with boundary:

$$\partial W_1 = (D^{m+1} \times S^n)_1 \cup (S^m \times D^{n+1})_3.$$

The intersection

$$(D^{m+1} \times S^n)_1 \cap (S^m \times D^{n+1})_3$$

of the two halves of the boundary is equal to $(S^m \times S^n)_1 = (S^m \times S^n)_3$. These two copies of $S^m \times S^n$ are identified under the composite correspondence

$$(x_1, y_1) \to (x_2, y_2) \to (x_3, y_3),$$

where

$$y_3 = y_2 = f_1(x_1) \cdot y_1, \qquad x_3 = f_2(y_3)^{-1} \cdot x_2 = f_2(y_3)^{-1} \cdot x_1.$$

But this is just the correspondence which was used to define the manifold $M(f_1, f_2)$. Thus ∂W_1 is homeomorphic to $M(f_1, f_2)$.

As it stands W_1 is not a differentiable manifold since there is an "angle" along the subset $(S^m \times S^n)_1$ of ∂W_1. Let W denote a differentiable manifold obtained by "straightening" this angle. (See the appendix to [7].) Clearly ∂W is diffeomorphic to $M(f_1, f_2)$.

2. The invariant $\lambda(M)$. First recall the index theorem of Hirzebruch [4]. If M_1 is a $4k$-manifold without boundary having Pontrjagin classes

$p_1, \cdots, p_k$, then the index $I(M_1)$ is equal to $L_k(p_1, \cdots, p_k)[M_1]$; where L_k is a certain polynomial.[3] For example

$$L_1 = p_1/3, \qquad L_2 = (7p_2 - p_1^2)/45, \cdots.$$

The coefficient s_k of p_k in L_k is particularly important. Hirzebruch expresses s_k in terms of the Bernoulli number B_k as follows (page 14):

$$s_k = 2^{2k}(2^{2k-1} - 1)B_k/(2k)!.$$

For example $s_2 = 7/45$, $s_3 = 62/945$, $s_4 = 127/4725$.

Now let M be a differentiable $(4k-1)$-manifold which

1) has the same rational homology groups as the $(4k-1)$-sphere,

and

2) is a boundary: $M = \partial W$ with W differentiable.[4] Then a rational number modulo 1,

$$\lambda(M) \in Q/Z,$$

is defined as follows. The natural homomorphism

$$j\colon H^i(W, M; Q) \to H^i(W; Q)$$

is an isomorphism for $0 < i < 4k-1$. Hence the Pontrjagin classes $p_1, \cdots, p_{k-1}$ of W can be lifted back to $H^*(W, M; Q)$. Define $\lambda(M)$ as the residue class of

$$(I(W) - L_k(j^{-1}p_1, \cdots, j^{-1}p_{k-1}, 0)[W])/s_k$$

modulo 1. (Here the symbol $[W]$ stands for the homomorphism $H^{4k}(W, M; Q) \to Q$ associated with the orientation of W; and $I(W)$ denotes the index of the quadratic form $\alpha \to (\alpha \cup \alpha)[W]$, where $\alpha \in H^{2k}(W, M; Q)$.)

LEMMA 2. *This residue class $\lambda(M)$ is an invariant of M: that is it does not depend on the choice of W.*

The proof is completely analogous to that in [5], [10] or [12]. If M is the boundary of both W_1 and W_2, then an unbounded $4k$-manifold M_1 is obtained from W_1, W_2 by

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[3] The symbol $[M_1]$ is used to denote the homomorphism of $H^{4k}(M_1; Q)$ into the rational numbers Q which is determined by an orientation for M_1. The index $I(M_1)$ is defined as the index of the quadratic form over $H^{2k}(M_1; Q)$ which is given by the formula $\alpha \to (\alpha \cup \alpha)[M_1]$.

[4] This second condition follows automatically if $H_*(M; Z)$ has no torsion. In fact every homology $(4k-1)$-sphere is a π-manifold (see [7]), and every π-manifold is a boundary (see [8]).

1) reversing the orientation of W_2;

2) matching W_1 and W_2 along the common boundary M;

3) constructing a differentiable structure in a neighborhood of $W_1 \cup W_2 = M$. (See [6] Lemma 4 or [7]). Then $I(M_1) = I(W_1) - (W_2)$; and each Pontrjagin number $p_{i_1} \cdots p_{i_r}[M_1]$ other than $p_k[M_1]$ is equal to the difference of corresponding Pontrjagin numbers for W_1, W_2. Now the index theorem for M_1 implies that the two definitions of $\lambda(M)$ differ only by the integer $p_k[M_1]$.

Example 1. For the $(4k-1)$-sphere it is clear that

$$\lambda(S^{4k-1}) \equiv 0 \pmod 1.$$

Example 2. For a 3-manifold the definition reduces to

$$\lambda(M^3) \equiv 3 \cdot I(W) \equiv 0 \pmod 1.$$

Example 3. For the 7-manifold M_3^7 of [5] the values

$$I(W) = 1, \quad (j^{-1}p_1)^2[W] = 36$$

give

$$\lambda(M_3^7) \equiv (45I(W) + (j^{-1}p_1)^2[W])/7 \equiv 4/7 \pmod 1.$$

Remark. If $H^*(M; Z)$ has no torsion, then the classes $j^{-1}p_i$ can be considered as integral cohomolgy classes, hence the Pontrjagin numbers of W are integers. This sharply restricts the denominator which $\lambda(M)$ can have. (For example $7\lambda(M^7)$ must be an integer.) On the other hand, if $H^*(M; Z)$ has torsion then arbitrarily large denominators may occur. (See the examples studied by Tamura.)

Example 4. In [7] § 4 certain homotopy spheres M_0^{4k-1} are constructed for $k > 1$. These have the property that $M_0^{4k-1} = \partial W$, where W is parallelizable, and $I(W) = 8$. Thus

$$\lambda(M_0^{4k-1}) \equiv 8/s_k \pmod 1.$$

For $k = 2$ this gives $\lambda(M_0^7) \equiv 3/7$ with denominator 7. For $k = 3, 4, 5, 6, 7$ the denominator of $\lambda(M_0^{4k-1})$ is 31, 127, 73, 1414477, and 8191 respectively. (These numbers are prime, except for $1414477 = 23 \cdot 89 \cdot 691$.) I do not know whether the inequality $8/s_k \not\equiv 0 \pmod 1$ holds for all $k > 1$.

In conclusion, the following three properties of the invariant λ are easily verified.

1) If the orientation of M is reversed, then λ changes sign.

2) For the connected sum of manifolds (see [7]), λ satisfies

$$\lambda(M_1 \# M_2) \equiv \lambda(M_1) + \lambda(M_2) \pmod 1.$$

3) λ is an invariant of the J-equivalence class of M. (See Thom [13] or [7].)

3. Computation of $\lambda(M(f_1, f_2))$. Define the *Pontrjagin homomorphism*

$$p_r\colon \pi_{4r-1}(SO_q) \to Z$$

as follows. Every map $f\colon S^{4r-1} \to SO_q$ induces a bundle ξ over S^{4r} with Pontrjagin class $p_r(\xi) \in H^{4r}(S^{4r}; Z) \approx Z$. Define $p_r(f)$ as the corresponding integer $p_r(\xi)[S^{4r}]$.

Let $f_1\colon S^m \to SO_{n+1}$, $f_2\colon S^n \to SO_{m+1}$ be arbitrary differentiable maps, with $m+n+1 = 4k-1$. First suppose that $m \neq n$.

LEMMA 3. *If $m \neq n$ then $M(f_1, f_2)$ is a topological sphere. The invariant $\lambda(M(f_1, f_2))$ is zero if m, n are not of the form $4r-1$. If $m = 4r-1$, $n = 4(k-r)-1$, then*

$$\lambda \equiv \pm\, p_r(f_1)\, p_{k-r}(f_2)\, s_r s_{k-r}/s_k \pmod 1.$$

Proof. We may assume that $m < n$. The exact sequence

$$\pi_m(SO_n) \to \pi_m(SO_{n+1}) \to \pi_m(S^n) = 0$$

implies that f_1 is homotopic to a map f_1' which carries S^m into the subset $SO_n \subset SO_{n+1}$. According to Lemma 1 the manifold $M(f_1', f_2)$ is homeomorphic to S^{m+n+1}. But it can be verified that $M(f_1, f_2)$ is homeomorphic to $M(f_1', f_2)$, and therefore is also homeomorphic to the sphere.

Next consider the manifold W constructed in Section 1. Recall that W is the union of a fibre bundle over S^{m+1} with fibre D^{n+1} and a fibre bundle over S^{n+1} with fibre D^{m+1}. Call these sets W_2 and W_3 respectively, Thus $W_2 \cup W_3$ is W and $W_2 \cap W_3$ is a topological cell.

These bundles have canonical cross-sections corresponding to the center point of the disk. Hence S^{m+1} and S^{n+1} are imbedded in W. It follows easily that W has the same homology groups as $S^{m+1} \vee S^{n+1}$ (the union with a single point in common). That is $H_i(W; Z)$ is infinite cyclic for i equal to 0, $m+1$, or $n+1$, and zero otherwise.

The homology intersection ring of W (see Lefschetz [14]) is described as follows. Let a and b stand for generators in dimensions $n+1$, $m+1$ respectively. Clearly a and b have intersection number ± 1. The self-inter-

sections $a \cdot a$ and $b \cdot b$ are zero. For example $a \cdot a$ is represented by a cycle of dimension

$$\dim a + \dim a - \dim W = n - m$$

which lies on the sphere $S^{n+1} \subset W_3$. Since $H_{n-m}(S^{n+1}; Z) = 0$, this cycle is homologous to zero.

Applying Poincaré duality it follows that $H^*(W, M; Z)$ *is free abelian on three generators, say* α *in dimension* $m+1$, β *in dimension* $n+1$, *and* $\alpha\beta$ *in dimension* $m+n+2$. *The cup products* $\alpha\alpha$ *and* $\beta\beta$ *are zero.* This implies that the index $I(W)$ is zero.

Computation of the Pontrjagin numbers of W. We may assume that $m = 4r-1$, $n = 4k-4r-1$. (If the dimensions are not of this form, then the Pontrjagin numbers are certainly zero, hence $\lambda \equiv 0$.) First consider the tangent bundle of W_2. This splits into a Whitney sum $\xi \oplus \eta$, where ξ is the bundle of vectors tangent to the fibre and η is the bundle of vectors normal to the fibre. Restricting η to the sphere $S^{m+1} \subset W_2$ we obtain the tangent bundle of S^{m+1} with trivial Pontrjagin classes. Restricting ξ to S^{m+1} we obtain the bundle determined by $(f_1) \in \pi_m(SO_{n+1})$. Thus $p_r(W_2) = p_r(\xi)$ is equal to the integer $p_r(f_1)$ multiplied by a generator of the infinite cyclic group $H^{4r}(W_2; Z)$. Using the isomorphisms

$$H^{4r}(W_2; Z) \leftarrow H^{4r}(W; Z) \xleftarrow{j} H^{4r}(W, M; Z)$$

it follows that

$$p_r(W) = \pm\, p_r(f_1)\, j(\alpha).$$

Similarly

$$p_{k-r}(W) = \pm\, p_{k-r}(f_2)\, j(\beta).$$

Thus *the Pontrjagin number* $(j^{-1}p_r)\,(j^{-1}p_{k-r})\,[W]$ *is equal to* $\pm\, p_r(f_1)\, p_{k-r}(f_2)$. All other Pontrjagin numbers of W are zero (except $(j^{-1}p_k)\,[W]$ which is not defined).

Computation of the coefficients of $p_r p_{k-r}$ in the Hirzebruch polynomial L_k. Define the symmetric function $\sum t_1^{i_1} \cdots t_a^{i_a}$ in indeterminates $t_1, \cdots, t_N$ as the sum of all monomials which can be obtained from $t_1^{i_1} \cdots t_a^{i_a}$ by permuting $t_1, \cdots, t_N$. Each possible monomial should be included only once in the sum. (For example $\sum t_1^r = t_1^r + \cdots + t_N^r$.) Hirzebruch showed[5] that the coefficient of $p_{i_1} \cdots p_{i_a}$ in L_k can be expressed in the form $\sum t_1^{i_1} \cdots t_a^{i_a}$, where $t_1, \cdots, t_N$ are certain fixed complex numbers. (Here N stands for

[5] See [4] § 1. 4. 1.

some fixed integer greater than or equal to k.) In particular, the coefficient s_k of p_k is equal to $\sum t_1{}^k$.

The product rule

$$(\sum t_1{}^r)(\sum t_1{}^{k-r}) = \sum t_1{}^k + \sum t_1{}^r t_2{}^{k-r} \qquad \text{for } r \neq k-r$$

is easily verified. Hence *the coefficient* $\sum t_1{}^r t_2{}^{k-r}$ *of* $p_r p_{k-r}$ *in* L_k *is equal to* $s_r s_{k-r} - s_k$.

Thus we have $I(W) = 0$ and

$$L_k(j^{-1}p_1, \cdots, j^{-1}p_{k-1}, 0)[W] = \pm p_r(f_1)\, p_{k-r}(f_2)\,(s_r s_{k-r} - s_k).$$

Dividing by s_k and reducing modulo one, this yields the required formula

$$\lambda(M) \equiv \pm p_r(f_1)\, p_{k-r}(f_2)\, s_r s_{k-r}/s_k \pmod 1$$

Now consider the case $m = n$. Again it is necessary to assume that m has the form $4r-1$ in order to obtain a non-trivial λ.

LEMMA 4. *If the maps* f_1, f_2 *both carry* S^m *into the subgroup* $SO_m \subset SO_{m+1}$, *then the formula*

$$\lambda(M) \equiv p_r(f_1)\, p_r(f_2)\, s_r s_r/s_{2r}$$

holds, just as in Lemma 3.

Proof. Just as above, $H_*(W;Z)$ is isomorphic to $H_*(S^{m+1} \vee S^{m+1})$. If $b, a \in H_{m+1}(W;Z)$ are the generators corresponding to the two spheres, then the intersection number $a \cdot b$ is ± 1. The hypothesis $f_1(S^m) \subset SO_m$ implies that the normal bundle of the first $(m+1)$-sphere in W has a cross-section. Hence the self-intersection number $a \cdot a$ is zero. Similarly $b \cdot b = 0$. It follows that W has index zero.

The computation of Pontrjagin classes for W proceeds as before. Thus

$$p_r(W) = \pm p_r(f_1)\, j\alpha \pm p_r(f_2)\, j\beta.$$

However the Pontrjagin number $(j^{-1}p_r)^2[W]$ is now equal to $\pm 2 p_r(f_1)\, p_r(f_2)$. On the other hand, the coefficient of $p_r p_r$ in L_{2r} is equal[5] to $\frac{1}{2}(s_r s_r - s_{2r})$. Thus the factor of $\frac{1}{2}$ cancels the 2, so that

$$\lambda(M(f_1 f_2)) \equiv \pm p_r(f_1)\, p_r(f_2)\, s_r s_r/s_{2r}$$

as before.

In order to make use of Lemmas 3, 4 it is necessary to know what integers $p_r(f)$ can occur.

THEOREM OF BOTT [2], [3]. *In the stable range $q \geqq 4r$ the Pontrjagin homomorphism*

$$p_r : \pi_{4r-1}(SO_q) \to Z$$

has image generated by

$$(2r-1)! \text{ if } r \text{ is even}$$

$$2(2r-1)! \text{ if } r \text{ is odd.}$$

For smaller values of q this result can be augmented as follows.

LEMMA 5. *If $q \leqq 2r$ then the homomorphism p_r is zero. If $q > 2r$ then p_r is non-zero. In fact there exists an element*

$$(f) \in \pi_{4r-1}(SO_q)$$

such that the prime factors of $p_r(f)$ are all less than $2r$.

Proof by descending induction on q. Suppose that the assertion has been proved for $q+1$, and that $q > 2r$. In the exact sequence

$$\pi_{4r-1}(SO_q) \to \pi_{4r-1}(SO_{q+1}) \to \pi_{4r-1}(S^q),$$

the third group is stable. According to Serre [9] a prime π can divide the order of this group only if $2\pi - 3$ is less than or equal to the difference $4r - q - 1$. The inequalities $2\pi - 3 \leqq 4r - q - 1$, $q > 2r$, yield $\pi \leqq r$. Thus any element of $\pi_{4r-1}(SO_{q+1})$, after being multiplied by primes less than or equal to r, can be lifted back to $\pi_{4r-1}(SO_q)$. This completes the induction.

If $q < 2r$, then the Pontrjagin class p_r of any SO_q-bundle is zero. If $q = 2r$, then $p_r(\xi^{2r})$ is the square of the Euler class of ξ^{2r}. (See Borel and Serre [1].) Since our base space is S^{4r}, this implies that $p_r(\xi^{2r}) = 0$; which completes the proof of Lemma 5.

Combining Lemmas 3, 4, 5 this proves:

THEOREM 1. *Suppose that r is an integer satisfying*

$$k/3 < r \leqq k/2.$$

Then there exists a differentiable manifold M homeomorphic to S^{4k-1} for which $\lambda(M)$ is congruent modulo 1 *to $s_r s_{k-r}/s_k$ times some integer with prime factors all less than $2(k-r)$.*

The proof is straightforward. (The inequality $k/3 < r$ guarantees the existence of a map $f_2 : S^{4(k-r)-1} \to SO_{4r}$ such that $p_{k-r}(f_2) \neq 0$.)

Note. Given k, the inequality $k/3 < r \leqq k/2$ has a solution r providing that $k = 2$ or $k \geqq 4$. It has no solution for $k = 1$ or 3.

THEOREM 2. *There exist at least 7 distinct differentiable structures on S^7; at least:*

$$\begin{aligned} 127 &\text{ on the 15-sphere,} \\ 73 &\text{ on the 19-sphere,} \\ 23 \cdot 89 \cdot 691 &\text{ on the 23-sphere,} \\ 8191 &\text{ on the 27-sphere, and at least} \\ 31 \cdot 151 \cdot 3617 &\text{ on the 31-sphere.} \end{aligned}$$

Proof. These results follow immediately from Theorem 1. As an example, for $k = 5$, taking $r = 2$, we have

$$s_2 s_3 / s_5 = 341/365.$$

Cancelling all prime factors less than 6 from the denominator, this leaves 73. But if M is a 19-manifold such that the denominator of $\lambda(M)$ is 73, then the first 73 manifolds

$$S^{19}, M, M \# M, M \# M \# M, \cdots$$

must be pairwise distinct. (Alternatively, if the homotopy class (f_1) is replaced by $q(f_1)$, $0 \leqq q < 73$, then we obtain 73 different values for the invariant λ.) Each one represent a possible differentiable structure for the 19-sphere.

In conclusion, here are two unsolved problems.

Problem 1. Does Theorem 1 imply the existence of non-standard differentiable structures on S^{4k-1} for all $k \geqq 4$? I have checked this only for k up to 14.

Problem 2. Is the invariant $\lambda(M^{4k-1})$ of a homotopy sphere always a multiple of the invariant

$$\lambda(M_0^{4k-1}) \equiv 8/s_k?$$

This question is of interest since, for any manifold M^{4k-1} which bounds a parallelizable manifold W, we have

$$\lambda(M^{4k-1}) \equiv I(W)/s_k \pmod 1,$$

and it can be shown that $I(W)$ is a multiple of 8. (Compare [7].)

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THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.*[1]

By BERTRAM KOSTANT.

1. Introduction.

1. Let $\mathfrak{g}$ be a complex simple Lie algebra and let G be the adjoint group of $\mathfrak{g}$. It is by now classical that the Poincaré polynomial $p_G(t)$ of G factors into the form,

$$p_G(t) = \prod_{i=1}^{l} (1 + t^{d_i}), \tag{1.1.1}$$

where l is the rank of $\mathfrak{g}$ and the d_i are odd integers. In this paper the integers m_i (elsewhere, sometimes $m_i + 1$) defined by $d_i = 2m_i + 1$ will be called the exponents of $\mathfrak{g}$. No doubt one of the reasons the problem of finding the exponents turned out to be as difficult as it was, is that there was no way known by which these numbers could be determined from a direct examination of the structure of $\mathfrak{g}$, particularly the root structure. The first procedure for extracting the exponents from the root structure of $\mathfrak{g}$ was found by R. Bott. The proof of the validity of this procedure depends upon Morse theory.[2] A second and much simpler way, which we shall presently describe, of "reading off" the exponents from the root structure of $\mathfrak{g}$ was discovered by Arnold Shapiro. (It is interesting that Shapiro discovered the procedure by misinterpreting the method of Bott.) However, even though one verifies that the numbers produced by this procedure agree with the exponents, unlike the case with Bott's method, the important question of proving that this "agreement" is more than just a coincidence remained open. The procedure is as follows: Let Δ^+ be the set of positive roots relative to some Weyl chamber in a Cartan subalgebra of $\mathfrak{g}$. If $\phi \in \Delta^+$ let $o(\phi)$ be the sum of the coefficients of ϕ relative to the basis of simple positive roots. Let b_k be the number of roots ϕ such that $o(\phi) = k$. Then $b_k - b_{k+1}$ is the number of times k occurs as an exponent of $\mathfrak{g}$. For example, if we apply this to the case where

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[2] See R. Bott, "An application of the Morse theory to the topology of Lie groups," *Bull. Soc. Math. France*, t. 84 (1956), pp. 251-281.

$\mathfrak{g}$ is Lie algebra of the special linear group $SL(n, C)$ then the fact that the number of matrix units e_{ij} such that $j-i=k$, where $1 \leqq k \leqq n-1$, is greater by one then the number of matrix units e_{ij} such that $j-i=k+1$ accounts for the fact that the exponents of $\mathfrak{g}$ are $1, 2, \cdots, n-1$.

After Shapiro informed us of this counting device for the exponents we observed that it could be reformulated as follows: The principal three-dimensional subalgebra $\mathfrak{a}_0$ of $\mathfrak{g}$ is a uniquely defined (up to conjugacy) three dimensional simple subalgebra (TDS) of $\mathfrak{g}$ which can be readily distinguished from other TDS by its properties. [It was discovered almost simultaneously by Dynkin and de Siebenthal (see [6] and [13]) and later used extensively by these authors (see e.g. [7])]. Using standard facts in the representation theory of a TDS it is not difficult to show that the observation of Shapiro is equivalent to the observation that if we decompose the adjoint representation of $\mathfrak{a}_0$ on $\mathfrak{g}$ into a direct sum of irreducible representations then the number of irreducible components is l and the dimensions of these components are d_i, $i=1, 2, \cdots, l$, where the d_i are given by (1.1.1). However, this reformulation of the procedure still does not supply a proof.

A second empirical procedure for finding the exponents was discovered by H. M. Coxeter. He recognized that the exponents can be obtained from a particular transformation γ in the Weyl group, which he had been studying, and which we take the liberty of calling a Coxeter-Killing transformation, in the following manner (see [5]): Let h be the order of γ. Coxeter observed that (1) h satisfies $hl=2r$, where r is the number of positive roots, (2) $m_i \leqq h$ for all i and (3) the eigenvalues of γ are ω^{m_i}, $i=1, 2, \cdots, l$, where $\omega = e^{2\pi i/h}$. A proof of (2) and (3) would provide, among other things, a proof of duality in the exponents m_i observed by Chevalley (see [3], p. 24) since non-real eigenvalues of γ necessarily occur in conjugate pairs. Requiring (1) $hl=2r$ as the only empirically observed fact such a proof was recently obtained by A. J. Coleman (see [4]). A proof that $hl=2r$ will be given in this paper. A second question posed in [4] of showing that $h=1+o(\psi)$, where ψ is the highest root, will also be settled here.

It will be the main result of this paper to establish a direct relationship between the principal TDS, its adjoint representation of $\mathfrak{g}$, and the transformation of Coxeter-Killing. The proof that the procedure of Shapiro yields the exponents is then a direct consequence of this relationship. A major role here is played by a particular conjugate class in G, the elements of which we call principal elements of G. It is shown that an element of G which induces a transformation of Coxeter-Killing on a Cartan subalgebra is necessarily a principal element. On the other hand a principal element of G

belongs to the subgroup corresponding to a principal TDS and is sufficiently specialized in that subgroup so that the adjoint representation of a principal TDS on $\mathfrak{g}$ is determined by its eigenvalues. That the conjugate class of principal elements is truly a distinguished one in G may be judged from the following characterization (one of several) of principal elements. Let A be an arbitrary regular element of G. Let k be the order of A (possibly ∞). Then $k \geqq h$ and $k = h$ if and only A is principal.

An important role is also played here by two distinguished classes of $\mathfrak{g}$, the elments of which we have, respectively, called principal nilpotent and cyclic (the latter name is derived from the transformation properties of such elements in the case when $\mathfrak{g}$ is the Lie algebra of $SL(n, C)$). As is the case with the principal elements of G these elements can be given simple characterizations (Corollary 5.3, Theorem 9.2 and Corollary 9.3).

In this paper §§ 1-4 are devoted to the theory of the general TDS. The main theorems here are Theorems 3.6 and 4.2. Both concern conjugacy questions. The first is an extension of a well known theorem of Jacobson-Morosov. A corollary of it puts the conjugate classes of TDS in $\mathfrak{g}$ in a canonical one-one correspondence with the conjugate classes of nilpotent elements in $\mathfrak{g}$. The second is implicit in the proof of a weaker theorem of Malcev.

In § 5 the theory of the principal TDS is taken up. For the most part the theorems here are devoted to characterizing principal TDS among all TDS and principal nilpotent elements among all nilpotent elements. The result here which is used most often in the remainder of the paper is Corollary 5.3.

The main results of the paper are given in §§ 6-9.

2. Preliminaries and the complex three dimensional simple Lie algebra.

1. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. Let n and l be respectively the dimension and rank of $\mathfrak{g}$. As usual the linear transformation

$$y \to [x, y]$$

is designated by $\operatorname{ad} x$ and $x \to \operatorname{ad} x$ is the adjoint representation of $\mathfrak{g}$ on itself. If $\mathfrak{u}$ is a Lie subalgebra of $\mathfrak{g}$ the mapping $x \to \operatorname{ad} x$ for $x \in \mathfrak{u}$ will be called the adjoint representation of $\mathfrak{u}$ on $\mathfrak{g}$. We distinguish two types of elements in $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is called nilpotent if $\operatorname{ad} x$ is nilpotent and is called semi-simple if $\operatorname{ad} x$ is completely reducible, that is, (since we are dealing with

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complex numbers) if $\operatorname{ad} x$ is diagonizable.[3] Only the zero element is both nilpotent and semi-simple.

We recall that any element x contained in a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is necessarily semi-simple and conversely every semi-simple element may be embedded in a Cartan subalgebra. (See e.g. [8], p. 119).

For any element $x \in \mathfrak{g}$ let $\mathfrak{g}^x$ designate the kernel of $\operatorname{ad} x$. We recall that x is called regular when x is semi-simple and $\mathfrak{g}^x$ is a Cartan subalgebra.

2.2. Let G designate the adjoint group of $\mathfrak{g}$. Since this is the only group associated with $\mathfrak{g}$ that we shall consider the operation of exponentiation (Exp) will be always understood to go from $\mathfrak{g}$ to G.

Elements x and y in $\mathfrak{g}$ are called conjugate if there exists $A \in G$ such that $Ax = y$.

2.3. The simplest complex semi-simple Lie algebra (up to isomorphism) is $\mathfrak{a}_1$, the Lie algebra of all complex 2×2 matrices of trace zero. In this case $n = 3$ and $l = 1$. One knows that any three dimensional complex semi-simple Lie algebra is isomorphic to $\mathfrak{a}_1$.

By conjugating any element of $\mathfrak{a}_1$ into Jordan canonical form the following is apparent:

(a) every non-zero element in $\mathfrak{a}_1$ is either semi-simple or nilpotent,

(b) the set of all non-zero nilpotent elements in $\mathfrak{a}_1$ form a single conjugate class,

(c) if $x, y \in \mathfrak{a}_1$ are semi-simple, $x \neq 0$, then y is conjugate to a unique, up to sign, scalar multiple of x. (Note also that the set of non-zero semi-simple elements coincides with the set of regular elements in $\mathfrak{a}_1$).

2.4. Let $\mathfrak{a}$ be a 3-dimensional complex simple Lie algebra. We recall further facts in the structure theory of $\mathfrak{a}$. Let $x \in \mathfrak{a}$ be a regular element. Then the eigenvalues of $\operatorname{ad} x$ are a, 0, $-a$ for some non-zero complex number a. We may modify x by scalar multiplication so that a and $-a$ take the values 1 and -1. This defines x uniquely up to conjugacy. We thus isolate a particular conjugate class in $\mathfrak{a}$. We thus isolate a particular conjugate class in $\mathfrak{a}$. The elements in this class will be called mono-semisimple.

Let $x \in \mathfrak{a}$ be a mono-semisimple element. Let $e_+ \in \mathfrak{a}$ be a non-zero eigen-

[3] It is not difficult to show that this very same definition [of semi-simple and nilpotent elements] may be achieved using any faithful representation of $\mathfrak{g}$ in place of the adjoint representation. This follows, e.g., from Lemma 5.4 and well known facts in the representation theory of $\mathfrak{g}$.

vector of $\operatorname{ad} x$ for the eigenvalue 1; e_+ is unique up to a non-zero scalar. Then e_-, an eigenvector of $\operatorname{ad} x$ for the value -1, is uniquely determined by condition $[e_+, e_-] = x$. One thus has the commutation relations

$$\text{(2.4.1)} \qquad \begin{aligned} &(1) \quad [x, e_+] = e_+ \\ &(2) \quad [x, e_-] = -e_- \\ &(3) \quad [e_+, e_-] = x \end{aligned}$$

for the basis x, e_+, e_- of $\mathfrak{a}$.

2.5. When $\mathfrak{a} = \mathfrak{a}_1$ the elements x, e_+, and e_- may be realized by the matrices

$$x = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad e_+ = 2^{-\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_- = 2^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For each positive integer d there exists up to equivalence one and only one linear irreducible representation of $\mathfrak{a}$ having that dimension. (For a complete treatment of the representation theory of $\mathfrak{a}_1$ see [12, Exposé no. 10].) To describe an irreducible representation π_d of $\mathfrak{g}$ on a d-dimensional vector space V first define the number k by the relation $d = 2k + 1$. It is of course to be noted that k is an integer if and only if d is odd. We may then find a basis v_j, $j = k, k-1, \cdots, -k$ of V, where the vectors are each unique up to a scalar factor, satisfying the following condition:

$$\pi_d(x) v_j = j \cdot v_j.$$

The behavior of $\pi_d(e_+)$ and $\pi_d(e_-)$ on the one dimensional spaces (v_j) is given by

$$\pi_d(e_+)(v_j) = (v_{j+1})$$
$$\pi_d(e_-)(v_j) = (v_{j-1})$$

where $v_{k+1} = v_{-k-1} = 0$. Thus v_j is an eigenvector for $\pi_d(e_+)\pi_d(e_-)$ and $\pi_d(e_-)\pi_d(e_+)$, where in fact

$$\pi_d(e_+)\pi_d(e_-)v_j = \tfrac{1}{2}(k - j + 1)(k + j)v_j$$

and

$$\pi_d(e_-)\pi_d(e_+)v_j = \tfrac{1}{2}(k - j)(k + j + 1)v_j.$$

It follows that with respect to the ordering $v_k, v_{k-1}, \cdots, v_{-k}$ of the basal elements, when the latter are suitably modified by scalar multiplication, one obtains for $\pi_d(x)$, $\pi_d(e_+)$ and $\pi_d(e_-)$ the matrices

$$\pi_d(x) = \begin{vmatrix} k & & & & 0 \\ & k-1 & & & \\ & & \ddots & & \\ & & & & \\ 0 & & & & -k \end{vmatrix}$$

$$\pi_d(e_+) = 2^{-\frac{1}{2}} \begin{vmatrix} 0 & (1(2k))^{\frac{1}{2}} & & & 0 \\ & 0 & (2(2k-1))^{\frac{1}{2}} & & \\ & & 0 & & \\ & & & \ddots & \\ & & & 0 & (2k(1))^{\frac{1}{2}} \\ 0 & & & & 0 \end{vmatrix}$$

$$\pi_d(e_-) = 2^{-\frac{1}{2}} \begin{vmatrix} 0 & & & & 0 \\ (1(2k))^{\frac{1}{2}} & 0 & & & \\ & (2(2k-1))^{\frac{1}{2}} & 0 & & \\ & & & \ddots & \\ & & & 0 & \\ 0 & & & (2k(1))^{\frac{1}{2}} & 0 \end{vmatrix}.$$

Several facts are to be noted. Among those we shall require are

(a) The dimension d of V is odd or even according as the eigenvalues of $\pi_d(x)$ are all integers or all half-integers.[4]

(b) The eigenvalues of $\pi_d(x)$ all occur with multiplicity 1 and the real number j is an eigenvalue of $\pi_d(x)$ if and only if $d-(2\,|\,j\,|+1)$ is a non-negative even integer.

(c) The number k, where $2k+1=d$, may be characterized as the highest eigenvalue of $\pi_d(x)$. Furthermore, the one-dimensional eigenspace for this eigenvalue k may be characterized as the kernel of $\pi_d(e_+)$.

Now assume π is an arbitrary, not necessarily irreducible, representation of $\mathfrak{a}$ on the finite-dimensional vector space V. One knows π may be decomposed into a direct sum of irreducible representations. It follows then that one knows π up to equivalence as soon as the dimensions of the irreducible components of π are given. That is, if n_k denotes the number of such components having dimension $2k+1$, then π is given when the sequence n_k, $k=0, \frac{1}{2}, 1, \cdots$ is known.

[4] We use the word half-integer to designate all numbers of the form $m+\frac{1}{2}$, where m is an integer.

(d) The problem of finding the sequence n_k can be reduced to an investigation of the kernel $W \subseteq V$ of $\pi(e_+)$. In fact, it follows from (c) that

$$\dim W = n_0 + n_1 + n_2 + \cdots.$$

Furthermore W is stable under $\pi(x)$ and if $w \in W$ is any eigenvector of $\pi(x)$, w may be embedded in an irreducible component of π. Hence if $k_1, k_2, \cdots, k_p$ are the eigenvalues of $\pi(x)$ on W, the dimensions of the irreducible components of π are respectively $2k_1+1, 2k_2+1, \cdots, 2k_p+1$. (Note that the k_i are non-negative.)

(e) The space V admits a canonical direct sum decomposition

$$V = V^E + V^O,$$

where V^E is spanned by eigenvectors of $\pi(x)$ belonging to half-integral eigenvalues and V^O is spanned by eigenvectors of $\pi(x)$ belonging to integral eigenvalues. It follows immediately from (a) that V^E and V^O are both stable subspaces for the representation π and that in the complete reduction of $\pi | V^E$ only irreducible representations of even dimension appear and the complete reduction of $\pi | V^O$ yields only irreducible representations of odd dimension.[5]

(f) Now let V_j be the eigenspace of $\pi(x)$ for the eigenvalue j. Clearly $\dim V_j = \dim V_{-j}$. Furthermore, if j is non-negative, it follows from (b) that

$$\dim V_j = n_j + n_{j+1} + n_{j+2} + \cdots.$$

The statement (f) has the following 2 consequences:

(g) The dimension of V_0, that is the dimension of the kernel of $\pi(x)$ (nullity of $\pi(x)$), equals $n_0 + n_1 + n_2 + \cdots$.

(h) If j is non-negative,

$$\dim V_j - \dim V_{j+1} = n_j.$$

Finally, we shall require

(i) If k is the maximal eigenvalue of $\pi(x)$,

$$V = \sum_{p=-2k}^{2k} V_{p/2}$$

is a direct sum and $\sum_{p=1}^{2k} V_{p/2}$ lies in the range of $\pi(e_+)$.

[5] If π is a representation on a vector space V and $U \subset V$ is stable under π, then $\pi | U$ denotes the representation on U obtained by restricting π to U.

3. Nilpotent elements and TDS.

1. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. Consider the question of determining all three dimensional simple subalgebras (TDS) in $\mathfrak{g}$. If $\mathfrak{a} \subseteq \mathfrak{g}$ is a TDS, then by considering the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ it follows from the representation theory of $\mathfrak{a}$ outlined in § 2.5 that any nilpotent element of $\mathfrak{a}$ is necessarily nilpotent in $\mathfrak{g}$ and any semi-simple element of $\mathfrak{a}$ is a semi-simple element of $\mathfrak{g}$ (Also see footnote 3). Since $\mathfrak{a}$ has only semi-simple and nilpotent elements (see § 2.3(a)), the question arises (1) which nilpotent and semi-simple elements of $\mathfrak{g}$ can be embedded in a TDS and (2) how does one find all such subalgebras. We shall first consider the case of nilpotent elements.

A theorem of Morosov asserts that every nilpotent element of $\mathfrak{g}$ can be embedded in a TDS. (See [11].) However, his proof was incomplete. Later in [9] Jacobson gave a correct proof of this result. Since the proof leads into Theorem 3.6, we shall give it here. The proof requires Lemma 3.3. With the exception of the proof of Lemma 3.3 the proof is the same as the one given by Jacobson.

3.2. A famous result of Jacobson asserts that if A and B are linear transformations on a finite dimensional space V with the condition that $[A, B]$ commutes with A, then $[A, B]$ is nilpotent. If in addition A is assumed nilpotent, the following lemma (which is no doubt known) asserts that AB is nilpotent.

LEMMA 3.2. *Let A and B be linear transformations on a finite dimensional space V. Assume A is nilpotent and*

$$[A, [A, B]] = 0.$$

Then AB is nilpotent.

Proof. Let V^k be the kernel of A^k. We wish to show AB leaves V^k invariant. The result is obvious if $k = 0$. Assume the result is known to be true for $k = r$. Let $x \in V^{r+1}$

$$ABx = [A, B]x + BAx.$$

Apply A^{r+1} to both sides. Then

$$\begin{aligned} A^{r+1}(AB)x &= [A, B]A^{r+1}x + A^{r+1}BAx \\ &= 0 + A^r(AB)Ax. \end{aligned}$$

But $Ax \in V^r$ and by our assumption $(AB)Ax \in V^r$. Thus $A^r(AB)Ax = 0$. Hence $ABx \in V^{r+1}$.

Assume AB is not nilpotent. Then there exists a scalar λ, $\lambda \neq 0$ and a vector $x \in V$, $x \neq 0$ such that

$$ABx = \lambda x. \tag{3.2.1}$$

Let k be the smallest integer such that $x \in V^{k+1}$.

Now

$$A^kABx = A^k[A,B]x + A^kBAx.$$

But $A^kB = BA^k + k[A,B]A^{k-1}$. Thus

$$\begin{aligned} A^kBAx &= BA^{k+1}x + k[A,B]A^kx \\ &= 0 + k[A,B]A^kx. \end{aligned}$$

Hence

$$A^kABx = (k+1)[A,B]A^kx.$$

But by (3.21)

$$A^kABx = \lambda A^kx.$$

Hence $[A,B]A^kx = (\lambda/k+1)A^kx$.

Since $A^kx \neq 0$, this contradicts Jacobson's lemma asserting the nilpotence of $[A,B]$. Q.E.D.

3.3. Now for any $x, y \in \mathfrak{g}$ let

$$(x,y) = \operatorname{tr} \operatorname{ad} x \operatorname{ad} y$$

be the Cartan-Killing bilinear form B on $\mathfrak{g}$. Using the non-singularity of B on $\mathfrak{g}$ we can now prove the following lemma. Lemma 3.3 is crucial in the proof that any nilpotent element $e \in \mathfrak{g}$ can be embedded in a TDS of $\mathfrak{g}$.

It is clear that if e is contained in a TDS then e must lie in the range of $(\operatorname{ad} e)^2$ since according to § 2.3(b), e can play the role of e_+ in the commutation relations (2.4.1). In particular, it is interesting enough to observe then that for any non-zero nilpotent element e, $(\operatorname{ad} e)^2 \neq 0$. (The degree of nilpotency of $\operatorname{ad} e$ is greater than 2.)

LEMMA 3.3. *Let $e \in \mathfrak{g}$ be a nilpotent element. Then e is in the range of $(\operatorname{ad} e)^2$.*

Proof. The invariance of B under the adjoint representation implies that, for any $z \in \mathfrak{g}$, $\operatorname{ad} z$ is skew-symmetric with respect to B and hence $(\operatorname{ad} z)^2$ is symmetric. In particular, this is true for $z = e$. But now if A is a symmetric operator (with respect to B) on $\mathfrak{g}$ and if R_A and K_A are, respectively, the range and kernel of A, then

$$(R_A, K_A) = 0$$

By the non-singularity of B, then, to show an element z lies in R_A, it suffices to show $(z, y) = 0$ for all $y \in K_A$. Letting $A = (\operatorname{ad} e)^2$, to prove the lemma it suffices to show that

$$[e, [e, y]] = 0 \tag{3.3.1}$$

implies

$$(e, y) = 0.$$

But under the adjoint representation (3.3.1) becomes

$$[\operatorname{ad} e, [\operatorname{ad} e, \operatorname{ad} y]] = 0.$$

Since $\operatorname{ad} e$ is nilpotent, we apply Lemma 3.2 to assert that $\operatorname{ad} e \operatorname{ad} y$ is nilpotent. But then by definition of B it is clear that $(e, y) = 0$. Q. E. D.

3.4. By Lemma 3.3, e may be written as

$$[[f, e], e] = e \tag{3.4.1}$$

for some $f \in \mathfrak{g}$. Let $x = [f, e]$ so that $[x, e] = e$.

LEMMA 3.4. *Let $\mathfrak{g}^e$ be the kernel of $\operatorname{ad} e$. Then $\mathfrak{g}^e$ is invariant under $\operatorname{ad} x$. Furthermore if m is the smallest integer such that $(\operatorname{ad} e)^{m+1} = 0$, then*

$$\prod_{p=0}^{m} (\operatorname{ad} x - p/2)$$

vanishes on $\mathfrak{g}^e$.

Proof. The space $(\operatorname{ad} e)^p \mathfrak{g}$ is the range of $(\operatorname{ad} e)^p$. We define a sequence of subspaces $\mathfrak{d}_p$ of $\mathfrak{g}^e$, $p = 0, 1, \cdots, m+1$, where

$$\mathfrak{g}^e = \mathfrak{d}_0 \subseteq \mathfrak{d}_1 \subseteq \cdots \subseteq \mathfrak{d}_{m+1} = 0, \tag{3.4.2}$$

by letting

$$\mathfrak{d}_p = (\operatorname{ad} e)^p \mathfrak{g} \cap \mathfrak{g}^p.$$

First observe that $\mathfrak{g}^e$ is invariant under $\operatorname{ad} x$. Indeed, if $y \in \mathfrak{g}^e$,

$$\begin{aligned} [e, [x, y]] &= [[e, x], y] \\ &= -[e, y] \\ &= 0. \end{aligned}$$

We now show that

$$(\operatorname{ad} x - p/2) : \mathfrak{d}_p \to \mathfrak{d}_{p+1}. \tag{3.4.3}$$

Let $y \in \mathfrak{d}_p$. Then $y = (\operatorname{ad} e)^p z$ for some $z \in \mathfrak{g}$. Now since $[x, e] = e$ clearly

$$[\operatorname{ad} x, (\operatorname{ad} e)^p] = p(\operatorname{ad} e)^p.$$

Thus

$$[\operatorname{ad} x, (\operatorname{ad} e)^p]z = py,$$

or

$$\begin{aligned}[x,y] - p\cdot y &= (\operatorname{ad} e)^p[x,z] \\ &= (\operatorname{ad} e)^p[[f,e]z] \\ &= (\operatorname{ad} e)^{p+1}[z,f] + (\operatorname{ad} e)^p \operatorname{ad} f[e,z].\end{aligned} \tag{3.4.4}$$

But

$$\begin{aligned}[(\operatorname{ad} e)^p, \operatorname{ad} f] &= -\sum_{i=0}^{p-1} (\operatorname{ad} e)^i \operatorname{ad} x (\operatorname{ad} e)^{p-1-i} \\ &= \tfrac{1}{2}p(p-1)(\operatorname{ad} e)^{p-1} - p \operatorname{ad} x (\operatorname{ad} e)^{p-1}.\end{aligned}$$

Applying this to $[e,z]$ we obtain

$$\begin{aligned}(\operatorname{ad} e)^p \operatorname{ad} f[e,z] &= \operatorname{ad} f(\operatorname{ad} e)^p[e,z] \\ &\quad + \tfrac{1}{2}p(p-1)(\operatorname{ad} e)^p z - p[x, (\operatorname{ad} e)^p z] \\ &= [f,[e,y]] + \tfrac{1}{2}p(p-1)y - p[x,y] \\ &= \tfrac{1}{2}p(p-1)y - p[x,y].\end{aligned}$$

But then (3.3.4) becomes

$$(p+1)[x,y] - \tfrac{1}{2}p(p+1)y = (\operatorname{ad} e)^{p+1}[z,f]$$

or

$$[x,y] - \tfrac{1}{2}py \in \mathfrak{d}_{p+1}.$$

This proves (3.4.3). It then follows immediately from (3.4.2) that

$$\prod_{p=0}^{m} (\operatorname{ad} x - p/2)$$

vanishes on $\mathfrak{g}^e$. Q. E. D.

It is an immediate consequence of Lemma 3.4 that $\operatorname{ad} x$ is completely reducible on $\mathfrak{g}^e$ and that its eigenvalues are restricted to non-negative integers and half-integers. In particular, what is essential for us at this point is

COROLLARY 3.4. *The linear transformation* $\operatorname{ad} x + 1$ *is non-singular on* $\mathfrak{g}^e$.

We can now prove

THEOREM 3.4 (Jacobson-Morosov). *Every nilpotent element of a complex semi-simple Lie algebra can be embedded in a* TDS.

Proof. Let e be nilpotent, $e \neq 0$, and let f and x be defined as in § 3.4.[6]

[6] The theorem holds when $e = 0$ once we know the existence of a single TDS. The existence of a TDS follows from the proof since g contains non-zero nilpotent elements.

In case $[x,f]=-f$ we would be done, that is, x, e, f would satisfy the desired commutation relations. The problem is to modify f so that this relation is satisfied without destroying the relation (3.4.1). Even if $[x,f]+f\neq 0$ we still have

$$[[x,f]+f,e]=0$$

as one easily checks. Thus $[x,f]+f\in \mathfrak{g}^e$. But now by Corollary 3.4, since $\operatorname{ad}x+1$ is non-singular on $\mathfrak{g}^e$, there exists a unique $g\in\mathfrak{g}^e$ such that

$$[x,f]+f=[x,g]+g.$$

Then writing e_+ for e and letting $e_-=f-g$ it follows that

$$\begin{aligned}[x,e_+]&=e_+\\ [x,e_-]&=-e_-\\ [e_+,e_-]&=x.\end{aligned}\tag{3.4.5}$$

This proves Theorem 3.4 as soon as one notes that x, e_+ and e_- must be linearly independent.

3.5. Any set of non-zero elements x, e_+ and e_- in $\mathfrak{g}$ satisfying the commutation relations (3.4.5) will henceforth be called an S-triple (to be written $\{x,e_+,e_-\}$). The element x will be called the neutral element of the S-triple and e_+ (resp. e_-) will be called the nil-positive (resp. nil-negative) element of the S-triple. It is obvious that the elements of an S-triple form a basis of a TDS. Two S-triples are called conjugate if there exists $A\in G$ which carries one set onto the other.

Given a non-zero nilpotent element $e\in\mathfrak{g}$ we wish now to find all S-triples which contain e as the nil-positive element. (By §2.3(b) this yields all TDS which contain e).

First we note the following corollary of the proof of Theorem 3.4.

COROLLARY 3.5. *Let $e\in\mathfrak{g}$ be nilpotent, $e\neq 0$, then x and e are respectively the neutral and nil-positive elements of an S-triple if and only if* (1) *x is in the range of* $\operatorname{ad}e$ *and* (2) $[x,e]=e$. *Furthermore if x and e satisfy these conditions such an S-triple system is unique (and hence x and e are contained in just one* TDS).

Proof. The first part of Corollary 3.5 follows from the proof of Theorem 3.4 and the definition of x used in the proof (see (3.4.1)). To prove the uniqueness of the nil-negative element assume that $\{x,e,f_1\}$ and $\{x,e,f_2\}$ are two S-triples. It is obvious then that

$$[e,f_1-f_2]=0$$

and hence $f_1 - f_2 \in \mathfrak{g}^e$. But

$$[x, f_1 - f_2] + f_1 - f_2 = 0$$

so that $f_1 - f_2$ is an eigenvector of $\operatorname{ad} x + 1$ on $\mathfrak{g}^e$. By this implies $f_1 = f_2$.
Q. E. D.

3.6. As a consequence of Corollary 3.5 the problem of finding all S-triple systems containing e as the nil-positive element reduces to finding all elements $x \in \mathfrak{g}$ which satisfy the conditions of Corollary 3.5. Towards this end define, for $e \in \mathfrak{g}$, the subspace

$$\mathfrak{g}_e = \operatorname{ad} e(\mathfrak{g}) \cap \mathfrak{g}^e,$$

the intersection of the range and kernel of $\operatorname{ad} e$. We now observe that $\mathfrak{g}_e$ is a Lie subalgebra of $\mathfrak{g}$. Indeed, $\mathfrak{g}^e$ is a Lie subalgebra of $\mathfrak{g}$. Therefore it sufficies only to show that $[u, v] \in \operatorname{ad} e(\mathfrak{g})$ if $u, v \in \mathfrak{g}_e$. Writing $v = [e, w]$ we have, since $u \in \mathfrak{g}^e$,

$$\begin{aligned}[u, v] &= [u, [e, w]] \\ &= [e, [u, w]].\end{aligned}$$

This proves $[u, v] \in \mathfrak{g}_e$. Now let G_e be the subgroup of G corresponding to the subalgebra $\mathfrak{g}_e$. Note that among other things the elements of G_e leave e fixed. We can now state

THEOREM 3.6. *Let $e \in \mathfrak{g}$, $e \neq 0$, be nilpotent. Let $\mathfrak{g}_e$ and G_e be as above. Then the elements of $\mathfrak{g}_e$ are all nilpotent (and hence the elements in G_e are unipotent). Let $x \in \mathfrak{g}$ be such that x and e are, respectively, the neutral and nil-positive elements of an S-*triple. *Then the linear coset $x + \mathfrak{g}_e$ of $\mathfrak{g}_e$ is the set of all neutral elements taken from all S-*triples *containing e as nil-positive element. Furthermore, any two elements in $x + \mathfrak{g}_e$ are conjugate. Moreover the conjugation can be performed by an element in G_e so that e is fixed under the conjugation. In fact for any $A \in G_e$, $Ax \in x + \mathfrak{g}_e$ and the map*

$$G_e \to x + \mathfrak{g}_e$$

defined by making A, $A \in G_e$, correspond to Ax is one-one and onto.

In other words (recalling Corollary 3.5*) if $\{x, e, f\}$ is the S-*triple *containing x and e, the map*

$$A \to \{Ax, e, Af\}$$

*sets up a one-one correspondence of the group G_e onto the set of all S-*triples *containing e as nil-positive element. Furthermore Ax ranges over $x + \mathfrak{g}_e$.*

Proof. Assume $\{x, e, f\}$ and $\{y, e, g\}$ are S-triples and e nil-positive in both cases while x and y are neutral.

Since

$$[x, e] = [y, e] = e,$$

it follows that $y - x \in \mathfrak{g}^e$. But clearly $[e, g - f] = y - x$. That is, $y - x \in \operatorname{ad} e(\mathfrak{g})$. Hence $y - x \in \mathfrak{g}_e$ or $y \in x + \mathfrak{g}_e$. Conversely, if $y \in x + \mathfrak{g}_e$, then clearly $[y, e] = e$ and $y \in \operatorname{ad} e(\mathfrak{g})$. Thus applying Corollary 3.5 y is a neutral element of an S-triple containing e.

Now according to Lemma 3.4 $\mathfrak{g}^e$ admits the direct decomposition

$$\mathfrak{g}^e = \sum_{p=0}^{m} \mathfrak{b}_{p/2},$$

where m is the smallest integer such that $(\operatorname{ad} e)^{m+1} = 0$ and $\mathfrak{b}_{p/2}$ is the eigenspace of $\operatorname{ad} x$ in $\mathfrak{g}^e$ belonging to the eigenvalue $p/2$. It is of course clear that

$$[\mathfrak{b}_{p/2}, \mathfrak{b}_{p'/2}] \subseteq \mathfrak{b}_{(p+p')/2}. \tag{3.6.1}$$

Let $\mathfrak{a}$ be the TDS spanned by x, e and f. Now if π_d is an irreducible representation of $\mathfrak{a}$ on a vector space of dimension d it is clear from § 2.5(i) that the eigenspace belonging to the highest eigenvalue $(\frac{1}{2}(d-1))$, of $\pi_d(x)$ lies in the range of $\pi_d(e)$ if and only if $d \geqq 2$. Now decompose $\mathfrak{g}$ into irreducible subspaces under the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ and apply this fact to the irreducible components. Recalling 2.5(d) it becomes clear then that the subspace $\mathfrak{g}_e$ of $\mathfrak{g}^e$ can be written

$$\mathfrak{g}_e = \sum_{p=1}^{m} \mathfrak{b}_{p/2} \tag{3.6.2}$$

and hence, in particular, $\operatorname{ad} x$ is non-singular on $\mathfrak{g}_e$.

One immediate observation from (3.6.1) and (3.6.2) is that $\mathfrak{g}_e$ is a nilpotent Lie subalgebra of $\mathfrak{g}$. Furthermore, since the eigenvalues of $\operatorname{ad} x$ on $\mathfrak{g}_e$ are strictly positive it is clear from a relation similar to (3.6.1) that $\operatorname{ad} w$ is nilpotent for every $w \in \mathfrak{g}_e$.[7] That is, the elements of $\operatorname{ad} \mathfrak{g}_e$ may be simultaneously triangulized with zeros appearing along the diagonal for every element. It follows then from well known facts concerning linear nilpotent Lie algebras that in such a case G_e is closed, simply connected—that is, G_e is homeomorphic to Euclidean space—and the exponential map

$$\operatorname{Exp}: \mathfrak{g}_e \to G_e$$

[7] That is, if $\mathfrak{g}_j$ is the eigenspace of $\operatorname{ad} x$ on $\mathfrak{g}$ for the eigenvalue j then clearly $[\mathfrak{g}_j, \mathfrak{b}_i] \subset \mathfrak{g}_{i+j}$.

is one-one and onto. That is, every $A \in G_e$ may be uniquely written $A = \operatorname{Exp} w$ for a unique $w \in \mathfrak{g}_e$. But then

$$Ax = x + [w, x] + \tfrac{1}{2}[w[w, x]] + \cdots. \tag{3.6.3}$$

Since $\mathfrak{g}_e$ is stable under $\operatorname{ad} x$ and since $\mathfrak{g}_e$ is a Lie subalgebra, all terms starting from the second on the right side of (3.6.3) lie in $\mathfrak{g}_e$ so that $Ax \in x + \mathfrak{g}_e$.

Now let $v \in \mathfrak{g}_e$. Assume that there exists a unique element $w_j \in \mathfrak{g}_e$ such that (1)

$$w_j \in \sum_{p=1}^{j} \mathfrak{b}_{p/2}$$

and (2)

$$\operatorname{Exp} w_j(x) - (x + v) \in \sum_{p=j+1}^{m} \mathfrak{b}_{p/2}$$

Now let z_{j+1} be the component of $\operatorname{Exp} w_j(x) - (x + v)$ in $\mathfrak{b}_{(j+1)/2}$. Then if

$$w_{j+1} = w_j + (2/j + 1) z_{j+1}$$

it is clear that $[w_{j+1}, x] = [w_j, x] - z_{j+1}$. On the other hand, for $i > 1$ it follows from (3.6.1) that the components of $(\operatorname{ad} w_{j+1})^i x$ and $(\operatorname{ad} w_j)^i x$ in $\mathfrak{b}_{s/2}$ are the same for all $s \leqq j + 1$. Thus

$$w_{j+1} \in \sum_{s=1}^{j+1} \mathfrak{b}_{s/2}$$

and

$$\operatorname{Exp} w_{j+1}(x) - (x + v) \in \sum_{s=j+2}^{m} \mathfrak{b}_{s/2}$$

and furthermore that in satisfying these conditions w_{j+1} is unique. If we define $w_1 = -2v_{\frac{1}{2}}$, where $v_{\frac{1}{2}}$ is the component of v in $\mathfrak{b}_{\frac{1}{2}}$, then w_1 uniquely satisfies (1) and (2) when $j = 1$. Thus we have proved inductively that there exists a unique $w \in \mathfrak{g}_e$ such that

$$\operatorname{Exp} w(x) = x + v. \qquad \text{Q. E. D.}$$

Note. *It is useful to observe that the proof, above, of the statement that Ax ranges over $x + \mathfrak{g}_e$ when A ranges over G_e depends essentially on just two facts, (1), the nilpotence of $\mathfrak{g}_e$ and (2), the non-singularity of* $\operatorname{ad} x$ *on* $\mathfrak{g}_e$.

We can now supplement Theorem 3.4 with

COROLLARY 3.6. *Let $e \in \mathfrak{g}$ be nilpotent, $e \neq 0$, and assume $e \in \mathfrak{a}_1 \cap \mathfrak{a}_2$, where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are two* TDS. *Then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate to each other. Furthermore, the conjugation can be chosen so as to leave e fixed.*

Proof. According to § 2.3(b) we may find two S-triples each containing e as nil-positive element and which, respectively, are bases for $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Corollary 3.6 then follows immediately from Theorem 3.6. Q.E.D.

3.7. Corollary 3.6 is actually just a special case of the following corollary. The set of all TDS in $\mathfrak{g}$ breaks up into conjugate classes under the action of G. Concerning these classes we have

COROLLARY 3.7. *The conjugate classes of* TDS *in* $\mathfrak{g}$ *are in a natural one-one correspondence with the conjugate classes of non-zero nilpotent elements in* $\mathfrak{g}$. *The correspondence is established by associating to the conjugate class of* $\mathfrak{a}$, *a* TDS *in* $\mathfrak{g}$, *the conjugate class of any non-zero nilpotent element in* $\mathfrak{a}$. *That is, two* TDS $\mathfrak{a}_1$ *and* $\mathfrak{a}_2$ *are conjugate if and only if* e_1 *and* e_2 *are conjugates, where* $e_1 \in \mathfrak{a}_1$, $e_2 \in \mathfrak{a}_2$ *and* e_1, e_2 *are non-zero nilpotent elements.*

Proof. Follows immediately from § 2.3(b), Theorem 3.4 and Corollary 3.6. Q.E.D.

4. Semi-simple elements and TDS.

1. We now consider the semi-simple elements of a TDS and take up questions of conjugacy. Let $\{x, e_+, e_-\}$ be an S-triple with x and e_+, respectively, as the neutral and nil-positive elements. We wish first to determine all S-triples which contain x (necessarily as neutral element). Let $\mathfrak{a}$ be the TDS spanned by x, e_+ and e_-. By considering the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ it follows from § 2.5 that the eigenvalues of $\operatorname{ad} x$ on $\mathfrak{g}$ are integers and half-integers. In fact, recalling § 2.5(f) and § 2.5(i), if $\mathfrak{g}_{p/2}$ is the eigenspace of $\operatorname{ad} x$ for the eigenvalue $p/2$ and if k is the maximal value of $\operatorname{ad} x$, then

$$\dim \mathfrak{g}_{p/2} = \dim \mathfrak{g}_{-p/2}$$

and

$$\mathfrak{g} = \sum_{p=-2k}^{2k} \mathfrak{g}_{p/2}.$$

In this case, however, we have the addtiional relation

$$[\mathfrak{g}_{p/2}, \mathfrak{g}_{p'/2}] \subseteq \mathfrak{g}_{(p+p')/2}. \tag{4.1.1}$$

Since we are concerned with S-triples containing x, interest focuses on $\mathfrak{g}_1$ since any nil-positive element in an S-triple containing x obviously belongs to $\mathfrak{g}_1$. In particular, $e_+ \in \mathfrak{g}_1$. The question arises which other elements of $\mathfrak{g}_1$

belong to S-triples containing x and whether such S-triples are conjugate to $\{x, e_+, e_-\}$. To settle this question we first consider $\mathfrak{g}_0 = \mathfrak{g}^x$.

It follows immediately from (4.1.1) that $\mathfrak{g}^x$ is a Lie subalgebra. Let G^x be the subgroup of the adjoint group corresponding to the subalgebra $\mathfrak{g}^x$. It also follows from (4.1.1) that each of the subspaces $\mathfrak{g}_{p/2}$ is stable under the adjoint representation of $\mathfrak{g}^x$ on $\mathfrak{g}$ and hence these spaces must be stable under G^x. We are particularly interested in the action of G^x on $\mathfrak{g}_1$.

4.2. Now for each element $e \in \mathfrak{g}_1$ it follows from (4.1.1) that

$$\operatorname{ad} e : \mathfrak{g}_0 \to \mathfrak{g}_1.$$

Let T_e be the restriction of $\operatorname{ad} e$ to $\mathfrak{g}_0$. Our interest now centers on what will be shown to be an important subset of $\mathfrak{g}_1$. Define

$$\hat{\mathfrak{g}}_1 = \{e \in \mathfrak{g}_1 \mid T_e \text{ maps } \mathfrak{g}_0 \text{ onto } \mathfrak{g}_1\}.$$

That is, $e \in \hat{\mathfrak{g}}_1$ if and only if the rank of T_e equals the dimensions of $\mathfrak{g}_1$.

We now observe

LEMMA 4.2A. *A necessary condition that an element $e \in \mathfrak{g}_1$ be the nil-positive element of an S-triple containing x is that $e \in \hat{\mathfrak{g}}_1$. In particular, $e_+ \in \hat{\mathfrak{g}}_1$.*

Proof. Indeed, assume e and x belong to an S-triple. Let $\mathfrak{a}'$ be the TDS which contains e and x. If we apply § 2.5(i) to the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ it follows that $\mathfrak{g}_1$ is the range of $\operatorname{ad} e$. On the other hand, it is clear from (4.1.1) that $\operatorname{ad} e(\mathfrak{g}) \cap \mathfrak{g}_1 = \operatorname{ad} e(\mathfrak{g}_0) \cap \mathfrak{g}_1$. Hence T_e must map $\mathfrak{g}_0$ onto $\mathfrak{g}_1$. Q. E. D.

The following topological properties of $\hat{\mathfrak{g}}_1$ are needed for the proof of Theorem 4.2 (Here one is inspired by the use of regular elements in the usual proof of the conjugacy of any two Cartan subalgebras.).

LEMMA 4.2B. *The set $\hat{\mathfrak{g}}_1$ is an open, dense and connected subset of $\mathfrak{g}_1$.*

Proof. It follows from Lemma 4.2A that $\hat{\mathfrak{g}}_1$ is not empty ($e_+ \in \hat{\mathfrak{g}}_1$). Choose a basis of $\mathfrak{g}_0$ and a basis of $\mathfrak{g}_1$. For any $e \in \mathfrak{g}_1$ set T_e^0 equal to the $\dim \mathfrak{g}_0 \times \dim \mathfrak{g}_1$ matrix determined by T_e and the given pair of bases. It is clear then that $\hat{\mathfrak{g}}_1$ is the set of all $e \in \mathfrak{g}_1$ such that at least one $\dim \mathfrak{g}_1 \times \dim \mathfrak{g}_1$ minor of T_e^0 is not zero. But it is an easily verified general fact that if F_j, $j = 1, 2, \cdots, m$, are m non-zero polynomials on a complex vector space V then the complement $\hat{V}$ to the set of common zeros in V of all the F_j is open, dense and connected. Indeed, if $u \in \hat{V}$ and $v \in V$, then there are only a finite

number of complex scalars λ such that $F_j^0(\lambda) = F_j(\lambda u + (1-\lambda)v)$ vanishes for at least one j. Q. E. D.

Now observe that $\hat{\mathfrak{g}}_1$ is invariant under G^x. The major point in the proof of Theorem 3.4 is the observation contained in the following lemma.

LEMMA 4.2C. *Let $e \in \hat{\mathfrak{g}}_1$; then the orbit $G^x e$ of e under the action of G^x is an open subset of $\mathfrak{g}_1$ (and hence of $\hat{\mathfrak{g}}_1$).*

Proof. The mapping $A \to Ae$ of G^x into $\mathfrak{g}_1$ is analytic. Thus it suffices to show that the differential of this mapping carries the tangent space to G^x at 1 *onto* the tangent space of $\mathfrak{g}_1$ at e. But the image of the former, under the differential, when translated to the origin of $\mathfrak{g}_1$ is just the subspace $\operatorname{ad} \mathfrak{g}^x(e)$ of $\mathfrak{g}_1$. But $\mathfrak{g}^x = \mathfrak{g}_0$ and since $e \in \hat{\mathfrak{g}}_1$, this space coincides with $\mathfrak{g}_1$, by definition of $\hat{\mathfrak{g}}_1$. Thus $G^x e$ is open in $\mathfrak{g}_1$. Q. E. D.

We have shown $G^x e$ is open in $\hat{\mathfrak{g}}_1$ for any $e \in \hat{\mathfrak{g}}_1$. But for $e_1, e_2 \in \mathfrak{g}_1$, $G^x e_1$ and $G^x e_2$ are the same sets or else they are disjoint. But this fact taken together with Lemma 4.2B (the latter asserting the connectivity of $\hat{\mathfrak{g}}_1$) implies that there can be at most one orbit. That is, $\hat{\mathfrak{g}}_1$ is itself a single orbit of G^x.

But then recalling Lemma 4.2A and observing that x is fixed under the action of G^x we see that the following theorem has been proved.

THEOREM 4.2. *Let $x \in \mathfrak{g}$ be the neutral element of an S-*triple $\{x, e_+, e_-\}$. *(See § 3.5.) As in § 2.1 let $\mathfrak{g}^x$ be the centralizer of x in $\mathfrak{g}$ and let G^x be the subgroup of G corresponding to $\mathfrak{g}^x$.*

Define

$$\mathfrak{g}_1 = \{e \in \mathfrak{g} \mid [x, e] = e\}.$$

Then

$$\operatorname{ad} e \colon \mathfrak{g}^x \to \mathfrak{g}_1 \tag{4.2.1}$$

for any $e \in \mathfrak{g}_1$.

*Let $e \in \mathfrak{g}$. Then e and x are, respectively, the nil-positive and neutral elements of an S-*triple *if and only if* (1) *$e \in \mathfrak{g}_1$ and* (2) *the map* (4.2.1) *is onto. Moreover, any two S-*triples *which contain x are conjugate to each other and the conjugation can be performed by an element in G^x.*

In other words, if

$$\hat{\mathfrak{g}}_1 = \{e \in \mathfrak{g}_1 \mid (4.2.1) \text{ is an onto map}\}$$

then $\hat{\mathfrak{g}}_1$ is the conjugate class of e_+ under the action of G^x. That is

$$\hat{\mathfrak{g}}_1 = G^x e_+.$$

Furthermore, $\hat{\mathfrak{g}}_1$ *coincides with the set of all nil-positive elements taken from all* S-triples *having* x *as neutral element.*

The statement of Theorem 4.2 gives a complete and simple description of the set of all nil-positive elements which "go" with a given neutral element. However, it is implicitly contained in Malcev's proof of the following corollary. Furthermore our proof of Theorem 4.2 although found without knowledge of Malcev's proof of Corollary 4.2 (yet with the knowledge that it had been proved) amounts to only a technical simplification of Malcev's proof.

Corollary 4.2 provides the basis by which all the TDS in any complex simple Lie algebra have been classified (see [7], § 8).

COROLLARY 4.2 (Malcev). *Two* TDS *in* $\mathfrak{g}$ *are conjugate if and only if any mono-semisimple element of one is conjugate to any mono-semisimple element of the other* (see [10]).

4.3. Now, continuing with the notation of § 4.1 and § 4.2, the subset $\hat{\mathfrak{g}}_1$ in $\mathfrak{g}_1$ is of course only a part of the conjugate class of e_+ in $\mathfrak{g}$. It does not seem likely that one can give a simple description of the entire class. Nevertheless, it is easy to describe a part of this class which is yet larger than $\hat{\mathfrak{g}}_1$ (see Theorem 4.3, Theorem 4.3 is required for the proof of Theorem 5.3) and we shall do this now.

For any $t = 0, 1, \cdots, 2k$ let

$$\mathfrak{n}_{t/2} = \sum_{p=t}^{2k} \mathfrak{g}_{p/2}.$$

It is clear from (4.1.1) that $\mathfrak{n}_{t/2}$ is a Lie subalgebra of $\mathfrak{g}$. Let $N_{t/2}$ be the subgroup of G corresponding to $\mathfrak{n}_{t/2}$. We now have, in terms if the preceding notation,

THEOREM 4.3. *The orbits* N_0x *and* N_0e_+ *of* e_+ *and* x *under the action of the group* N_0 *are as follows:*

$$N_0x = x + \mathfrak{n}_{\frac{1}{2}}$$

and

$$N_0e_+ = \hat{\mathfrak{g}}_1 + \mathfrak{n}_{\frac{3}{2}}.$$

In fact, G^x *is a subgroup of* N_0 *and* $N_{\frac{1}{2}}$ *is a normal subgroup of* N_0. *Moreover, the elements of* $N_{\frac{1}{2}}$ *are unipotent linear transformations of* $\mathfrak{g}$. *Furthermore,* N_0 *can be written as a semi-direct product*

$$N_0 = N_{\frac{1}{2}}G^x$$

(every element $C \in N_0$ is uniquely written $C = AA'$, where $A \in N_{\frac{1}{2}}$, $A' \in G^x$) and the correspondence

$$A \to Ax$$

sets up a one-one mapping of $N_{\frac{1}{2}}$ onto $x + \mathfrak{n}_{\frac{1}{2}}$ while the correspondence

$$A \to Ae_+$$

defines a mapping of N_0 onto $e_+ + \mathfrak{n}_{\frac{3}{2}}$.

Proof. Since $\mathfrak{n}_0 = \mathfrak{g}^x + \mathfrak{n}_{\frac{1}{2}}$ is a semi-direct sum (that is, $[\mathfrak{g}^x, \mathfrak{n}_{\frac{1}{2}}] \subseteq \mathfrak{n}_{\frac{1}{2}}$) it follows that every element $C \in N_0$ can be written, $C = AA'$, where $A \in N_{\frac{1}{2}}$ and $A' \in G^x$. On the other hand, if $A \in N_{\frac{1}{2}}$ and $y \in \mathfrak{g}_{t/2}$, then $Ay = y + w$, where $w \in \sum_{p=t+1}^{2k} \mathfrak{g}_{p/2}$. This implies A is unipotent. It also implies that $N_{\frac{1}{2}} \cap G^x = (1)$ and hence N_0 is a semi-direct product of $N_{\frac{1}{2}}$ and G^x. Now observe that $\mathfrak{n}_{\frac{1}{2}}$ is stable under $\operatorname{ad} x$ and $\operatorname{ad} x$ is non-singular on $\mathfrak{n}_{\frac{1}{2}}$. The proof that the correspondence $A \to Ax$ sets up a one-one mapping of $N_{\frac{1}{2}}$ onto $x + \mathfrak{n}_{\frac{1}{2}}$ then proceeds in essentially the same way as the proof of Theorem 3.6 (Recall that $[\mathfrak{n}_{t/2}, \mathfrak{n}_{t'/2}] \subseteq \mathfrak{n}_{(t+t')/2}$ and see the note following Theorem 3.6) and we shall not repeat it. On the other hand, one cannot claim that the mapping of $N_{\frac{1}{2}}$ into $e_+ + \mathfrak{n}_{\frac{3}{2}}$ defined by the correspondence $A \to Ae$ is one-one. This is because $\operatorname{ad} e_+$ annihilates non-zero elements in $\mathfrak{n}_{\frac{1}{2}}$. But $\operatorname{ad} e_+$ maps $\mathfrak{g}_{p/2}$ into $\mathfrak{g}_{(p/2)+1}$ and applying 2.5(i) to the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ it follows that

$$\operatorname{ad} e_+ : \mathfrak{n}_{\frac{1}{2}} \to \mathfrak{n}_{\frac{3}{2}}$$

is an onto mapping. Proceeding then in a manner similar to the proof of Theorem 3.6 it follows easily that $N_{\frac{1}{2}}$ is mapped onto $e_+ + \mathfrak{n}_{\frac{3}{2}}$ by the correspondence $A \to Ae_+$. Q. E. D.

As we have remarked, Corollary 4.2 has provided the basis by which the conjugate classes of TDS have been classified (See [7], § 8). That is if x is a mono-semisimple element of a TDS, we know that x is a semi-simple element of $\mathfrak{g}$ and that the eigenvalues of $\operatorname{ad} x$ are real (in fact are integers and half-integers). To classify the conjugate classes of TDS it suffices then by Corollary 4.2 to find a fundamental domain for the action of G on the set of semi-simple x in $\mathfrak{g}$ such that $\operatorname{ad} x$ has real eigenvalues and determine which in the domain are mono-semisimple elements of TDS. The Weyl chamber is such a domain and we shall presently consider it.

(In view of Corollary 3.7 for the purposes of classifying conjugate classes of TDS it might be well to look for natural representative elements for the

conjugate classes of nilpotent elements. To my knowledge no such investigation has been undertaken).

5. The principal TDS.

1. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra which we shall assume is fixed once and for all. Let Δ be the set of roots with respect to $\mathfrak{h}$ and let e_ϕ, $\phi \in \Delta$ be representative root vectors so that we have usual direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\phi \in \Delta} (e_\phi).$$

Since B is non-isngular on $\mathfrak{h}$, we may assume Δ is embedded in $\mathfrak{h}$. That is, if $\phi \in \Delta$, then ϕ may be identified with an element in $\mathfrak{h}$ by the relation

$$[x, e_\phi] = (x, \phi) e_\phi.$$

for all $x \in \mathfrak{h}$. One knows then if e_ϕ and $e_{-\phi}$ are normalized only in so far as

$$(e_\phi, e_{-\phi}) = 1, \tag{5.1.0}$$

as we shall assume, then

$$[e_\phi, e_{-\phi}] = \phi. \tag{5.1.1}$$

Now let $\mathfrak{h}^\#$ be the real linear space in $\mathfrak{h}$ spanned by the roots. One knows that

$$\mathfrak{h} = \mathfrak{h}^\# + i\mathfrak{h}^\#$$

is a real direct sum. We recall that B is positive definite on $\mathfrak{h}^\#$ (see e. g. [12], p. 10-04). In particular then, (x, ϕ) is real for all $x \in \mathfrak{h}^\#$ at all $\phi \in \Delta$. This means $\mathfrak{h}^\#$ can be characterized as the set of all elements $x \in \mathfrak{h}$ such that $\operatorname{ad} x$ has real eigenvalues.

Hereafter, if Y is any subset of $\mathfrak{h}$, we will let $Y^\# = Y \cap \mathfrak{h}^\#$.

Now for any $\phi \in \Delta$ let

$$\mathfrak{h}[\phi] = \{x \in \mathfrak{h} \mid (\phi, x) = 0\}.$$

Then, clearly, if R is the set of all regular elements in $\mathfrak{h}$,

$$R^\# = \mathfrak{h} - \bigcup_{\phi \in \Delta} \mathfrak{h}[\phi].$$

The connected components of $R^\#$ are called open Weyl chambers. Now assume a lexicographical ordering is given in $\mathfrak{h}^\#$. Let Δ^+ (resp. Δ^-) be the set of positive roots (resp. negative roots). The ordering distinguishes a particular open Weyl chamber D^0 which can be defined by

$$D^0 = \{x \in \mathfrak{h}^\# \mid (\phi, x) > 0 \text{ for all } \phi \in \Delta^+\}.$$

The closure D of D^0 is called a Weyl chamber. It can be defined in the same way as D^0 except that one replaces the strict inequality $>$ by the inequality $\geqq$. The set D is a fundamental domain for the action of G on the set of all semi-simple elements y in $\mathfrak{g}$ such that $\operatorname{ad} y$ has real eigenvalues. That is, if y is such an element there exists one and only one ([12], p. 16-08 and [7], Lemma 8.2) element x in D which is conjugate to y.

Now let $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ be the set of simple positive roots. The simple positive roots form a basis of $\mathfrak{h}$ and for every root ϕ, upon writing

$$\phi = \sum_{i=1}^{l} n_i \alpha_i,$$

one knows that the coefficients n_i are integers which are all non-negative or all non-positive according as $\phi \in \Delta^+$ or $\phi \in \Delta^-$. We define the order $o(\phi)$ of ϕ by letting

$$o(\phi) = \sum_{i=1}^{l} n_i. \tag{5.1.2}$$

Obviously, if ϕ_1, ϕ_2 and $\phi_1 + \phi_2 \in \Delta$, then

$$o(\phi_1 + \phi_2) = o(\phi_1) + o(\phi_2). \tag{5.1.3}$$

Let ϵ_i, $i = 1, 2, \cdots, l$, be the dual basis in h to the α_j. Since

$$(\epsilon_i, \alpha_j) = \delta_{ij}, \tag{5.1.4}$$

it is clear that $\epsilon_i \in D$. More generally, if

$$x = \sum_{i=1}^{l} a_i \epsilon_i, \tag{5.1.5}$$

then $x \in D$ if and only if $a_i \geqq 0$, $i = 1, 2, \cdots, l$. Indeed,

$$(x, \alpha_i) = a_i. \tag{5.1.6}$$

Now assume $x \in D$ is the neutral element of an S-triple $\{x, e_+, e_-\}$. Since the eigenvalues of $\operatorname{ad} x$ are integral multiples of $\frac{1}{2}$, it follows from that in the expansion (5.1.5) $a_i = \frac{1}{2} m_i$ for some non-negative integer m_i. On the other hand, it is not possible for a_i to be greater than 1. Indeed, since $\operatorname{ad} x(e_-) = -e_-$, it is clear that

$$e_- = \sum_{\phi \in \Delta^-} c_\phi e_\phi,$$

where we emphasize the summation is over negative roots. Hence it follows from § 2.5(d) (with signs reversed. Also see Lemma 3.4) that if $a_i > 1$, $\operatorname{ad} e_-(e_{\alpha_i})$ is a non-zero eigenvector of $\operatorname{ad} x$ with the positive eigenvalue $a_i - 1$.

But then since $x \in D$, this implies $[e_-, e_{\alpha_i}]$ is a sum of root vectors for positive roots. This, however, clearly contradicts the simplicity of α_i. We have proved then that $a_i = 0$, $\frac{1}{2}$, or 1 for $i = 1, 2, \cdots, l$. More than this we have (see [7], Theorem 8.3).

LEMMA 5.1. *Let the Weyl chamber D and the basis ϵ_i, $i = 1, 2, \cdots, l$ of $\mathfrak{h}$ be given as above. Assume that $x \in D$ is a mono-semisimple element of a* TDS. *Then*

$$x = \sum_{i=1}^{l} a_i \epsilon_i,$$

where for each $i = 1, 2, \cdots, l$, we have $a_i = 0$, $\frac{1}{2}$, or 1. Furthermore, if $x_1, x_2, \cdots, x_b \in D$ is the set (ordered) of all elements in D which happen to be mono-semisimple elements in at least one TDS *(so that $b < 3^l$), then there are exactly b conjugate classes of* TDS *in $\mathfrak{g}$. In fact, if x_i is a mono-semisimple element of the* TDS $\mathfrak{a}_i$ *for $i = 1, 2, \cdots, b$, then the subalgebras $\mathfrak{a}_i$, $i = 1, 2, \cdots, b$, are representatives of the conjugate classes.*

For each simple Lie algebra $\mathfrak{g}$ Dynkin lists [7] which among the $3^l - 1$ choices are indeed mono-semisimple elements of a TDS.

5.2. We retain the notation of Lemma 5.1. Among the elements x_j, $j = 1, 2, \cdots, b$ there is a distinguished one. This is the case when $a_i = 1$, $i = 1, 2, \cdots, l$. That is, for one of the x_j all the a_i (in (5.15)) take the maximal possible value.

LEMMA 5.2. *Let D and ϵ_i, $i = 1, 2, \cdots, l$, be as defined in* 5.1. *Let $x_0 \in D$ be given by*

$$x_0 = \sum_{i=1}^{l} \epsilon_i. \tag{5.2.1}$$

Then x_0 is a mono-semisimple element of a TDS

Proof. Since the elements of α_i form a basis of $\mathfrak{h}^\#$, x_0 may be uniquely written

$$x_0 = \sum_{i=1}^{l} r_i \alpha_i.$$

Now let c_i, $i = 1, 2, \cdots, l$, be any l non-zero complex numbers.

Define

$$e_0 = \sum_{i=1}^{l} c_i e_{\alpha_i} \tag{5.2.2}$$

and

$$f_0 = \sum_{i=1}^{l} (r_i / c_i) e_{-\alpha_i}. \tag{5.2.3}$$

Clearly $(\alpha_i, x_0) = 1$ for all i. Thus

$$[x_0, e_0] = e_0$$

and

$$[x_0 f_0] = -f_0.$$

On the other hand, since $\alpha_i - \alpha_j$ is never a root, it follows from (5.1.1) that

$$[e_0, f_0] = x_0$$

so that $\{x_0, e_0, f_0\}$ form a S-triple. Q.E.D.

Let x_0, e_0 and f_0 be as defined in the proof of Lemma 5.2. Let $\mathfrak{a}_0$ be the TDS spanned by the vectors. The TDS $\mathfrak{a}_0$ or any conjugate TDS is called (originally in [6]) a principal TDS of $\mathfrak{g}$. It and some of its properties were discovered by Dynkin and de Siebenthal (See [6] and [13]). We shall call the S-triple (or any of its conjugates) $\{x_0, e_0, f_0\}$ a principal S-triple. The matrices exhibited in § 2.5 are an example of a principal S-triple in the Lie algebra of $SL(d, C)$.

One of the first properties we observe about the neutral element of a principal S-triple is that it is regular. In fact, $x_0 \in \mathfrak{h}^\#$ and for any $\phi \in \Delta$ clearly

$$(x_0, \phi) = o(\phi) \neq 0 \tag{5.2.4}$$

(see (5.1.2)). We shall call x_0 or any element of $\mathfrak{g}$ conjugate to x_0 a principal regular element of $\mathfrak{g}$. Any element of the conjugate class of e_0 will be called a principal nilpotent element of $\mathfrak{g}$.

One of the most significant ways in which a principal TDS is distinguished among all TDS is in regard to its adjoint representation on $\mathfrak{g}$.

For any TDS $\mathfrak{a}$ of $\mathfrak{g}$ let $n(\mathfrak{a})$, $n^E(\mathfrak{a})$ and $n^O(\mathfrak{a})$ designate, respectively, the number of irreducible components occurring in the complete reduction of the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$, the number having even dimension and the number having odd dimension.

THEOREM 5.2. *Let $\mathfrak{a}$ be any* TDS *of $\mathfrak{g}$. Then $n(\mathfrak{a}) \geqq l$. Furthermore $\mathfrak{a}$ is principal if and only if $n(\mathfrak{a}) = l$.*

Proof. Let $\{x, e, f\}$ be an S-triple whose linear span is $\mathfrak{a}$. In fact, employing the notation of § 5.1 we may assume $x \in D \subseteq \mathfrak{h}$. Since $\mathfrak{h}$ is contained in $\mathfrak{g}^x$ (see § 2.1), obviously $\dim \mathfrak{g}^x \geqq l$. But now according to 2.5(g) we have $n^O(\mathfrak{a}) = \dim \mathfrak{g}^x$. Thus

$$n(\mathfrak{a}) \geqq n^O(\mathfrak{a}) = \dim \mathfrak{g}^x \geqq l \tag{5.2.5}$$

which proves the first part of Theorem 5.2. Now if $\mathfrak{a}$ is principal x is regular and hence $\dim \mathfrak{g}^x = l$. On the other hand, in this case all the eigenvalues of $\operatorname{ad} x$ are integers (see (5.1.2)) so that $n^E(\mathfrak{a}) = 0$ (see §2.5(e)). Thus if $\mathfrak{a}$ is principal, $n(\mathfrak{a}) = n^O(\mathfrak{a}) = l$. Now, conversely, assume $n(\mathfrak{a}) = l$. But then by (5.2.5) $\dim \mathfrak{g}^x = l$ which implies x is regular. Writing $x = \sum_{i=1}^{l} a_i \epsilon_i$ as in §5.1, it follows then that $a_i = \frac{1}{2}$ or 1 for all i. But since $n^O(\mathfrak{a}) = n(\mathfrak{a})$, that is, $n^E(\mathfrak{a}) = 0$, this means (see §2.5(e)) the eigenvalues of $\operatorname{ad} x$ are all integral. Thus $a_i \neq \frac{1}{2}$ by (5.1.6) and hence $x = \sum_{i=1}^{l} \epsilon_i$ so that $\mathfrak{a}$ is principal by Corollary 4.2. Q. E. D.

As a corollary of the proof we have

COROLLARY 5.2. *Let $\mathfrak{a}$ be any* TDS *of $\mathfrak{g}$; then $n^O(\mathfrak{a}) \geqq l$. Furthermore, $n(\mathfrak{a})^O = l$ if and only if the mono-semisimple elements of $\mathfrak{a}$ are regular in $\mathfrak{g}$. In case $\mathfrak{a}$ is principal $n^E(\mathfrak{a}) = 0$.*

5.3. The following corollary of Theorem 5.2 distinguishes the principal nilpotent elements in the set of all nilpotent elements in $\mathfrak{g}$. The proof follows immediately from §2.5(d) and Theorem 5.2.

COROLLARY 5.3. *Let e be a nilpotent element in $\mathfrak{g}$; then* $\dim \mathfrak{g}^e \geqq l$ *and* $\dim \mathfrak{g}^e = l$ *if and only if e is principal nilpotent.*

Now let $\{x_0, e_0, f_0\}$ be the principal S-triple defined as in §5.2. We will now apply the theory of §4.1 to this S-triple.

First of all, since the eigenvalues of $\operatorname{ad} x_0$ are integral, we observe that $\mathfrak{g}_{p/2} = 0$ whenever $p/2$ is not an integer and for j a non-zero integer

$$\mathfrak{g}_j = \sum_{o(\phi)=j} (e_\phi) \tag{5.3.1}$$

and

$$\mathfrak{g}_0 = \mathfrak{h}.$$

We will write $\mathfrak{n}$ for $\mathfrak{n}_{\frac{1}{2}} = \mathfrak{n}_1$ and $\mathfrak{s}$ for $\mathfrak{n}_0$. Then

$$\mathfrak{n} = \sum_{\phi \in \Delta^+} (e_\phi)$$

(as is well known) is a maximal Lie subalgebra of nilpotent elements and $\mathfrak{s} = \mathfrak{h} + \mathfrak{n}$ is a maximal solvable Lie subalgebra of $\mathfrak{g}$. Let H, N and S be the subgroups of G corresponding to $\mathfrak{h}$, $\mathfrak{n}$ and $\mathfrak{s}$.

Recalling that

$$e_0 = \sum_{i=1}^{l} c_i e_{\alpha_i},$$

where $c_i \neq 0$, $i = 1, 2, \cdots, l$ it follows from Theorems 4.2 and 4.3 that in this case

$$He_0 = \hat{\mathfrak{g}}_1 = \{e \in \mathfrak{g}_1 \mid e = \sum_{i=1}^{l} a_{\alpha_i} e_{\alpha_i}, \text{ where } a_{\alpha_i} \neq 0,\ i = 1, 2, \cdots, l\}$$

and (since $\mathfrak{n}_{\frac{1}{2}} = \mathfrak{n}_2$)

$$Se_0 = \hat{\mathfrak{g}}_1 + \mathfrak{n}_2 = \{e \in \mathfrak{n} \mid e = \sum_{\phi \in \Delta^+} a_\phi e_\phi, \text{ where } a_\phi \neq 0 \text{ when } \phi = \alpha_i,\ i = 1, 2, \cdots, l\}. \tag{5.3.2}$$

We emphasize that (5.3.2) above implies that "almost all" the elements in the maximal subalgebra of nilpotent elements $\mathfrak{n}$ are principal nilpotent (and hence lie in a single conjugate class).

We wish to prove now that Se_0 contains every principal nilpotent in $\mathfrak{n}$, that is, we have the following simple characterization of those elements in $\mathfrak{n}$ which are principal nilpotent.

THEOREM 5.3. *Let $\mathfrak{n} \subseteq \mathfrak{g}$ be the maximal Lie subalgebra of nilpotent elements given by*

$$\mathfrak{n} = \sum_{\phi \in \Delta^+} (e_\phi).$$

Let $e \in \mathfrak{n}$,

$$e = \sum_{\phi \in \Delta^+} a_\phi e_\phi$$

be arbitrary. Then e is principal nilpotent if and only if $a_\phi \neq 0$ for $\phi = \alpha_i$, $i = 1, 2, \cdots, l$.

Proof. By (5.3.2) if e satisfies the condition stated in Theorem 5.3, then e is principal nilpotent. Conversely, let $e \in \mathfrak{n}$ be principal nilpotent and assume $a_{\alpha_j} = 0$, for some j.

Assume first that $\mathfrak{g}$ is simple. Let $\psi \in \Delta$ be the highest root. We recall two basic facts about the highest root. One, upon writing $\psi = \sum_{i=1}^{l} q_i \alpha_i$ the integers q_i satisfy

$$q_i \geqq 1 \tag{5.3.3}$$

and

$$o(\psi) > o(\phi) \tag{5.3.4}$$

for any $\phi \in \Delta$, $\phi \neq \psi$. Indeed, both of these facts are immediate consequences of the following single fact: If $\phi = \sum_{i=1}^{l} t_i \alpha_i$ is any root, then

$$t_i \leqq q_i \tag{5.3.5}$$

for $i = 1, 2, \cdots, l$. Finally, (5.3.5) is a consequence of well known facts in representation theory; in fact, the adjoint representation of $\mathfrak{g}$ is irreducible and by definition of ψ, e_ψ is the weight vector belonging to the highest weight, ψ. One knows then that e_ϕ, for any $\phi \in \Delta$, is obtained by applying polynomials (non-commutative) in the operators $\operatorname{ad} e_{-\alpha_i}$, $i = 1, 2, \cdots, l$, to e_ψ. This proves (5.3.5). Let

$$q = \sum_{i=1}^{l} q_i = o(\psi).$$

Now let $\mathfrak{a}_0$ be as in § 5.2 and consider the adjoint representation of $\mathfrak{a}_0$ on $\mathfrak{g}$. It follows immediately from § 2.5(d) and (5.3.4) that (1) e_ψ lies in an irreducible component $\mathfrak{b} \subseteq \mathfrak{g}$, (2) $\dim \mathfrak{b} = 2q + 1$, (3) $e_{-\psi} \in \mathfrak{b}$ and (4) any other irreducible has dimension less than $\dim \mathfrak{b}$. A direct consequence of these facts is that

$$(\operatorname{ad} e_0)^{2q+1} = 0$$

and

$$(\operatorname{ad} e_0)^{2q}(e_{-\psi}) = a e_\psi$$

where $a \neq 0$.

Now since e is conjugate to e_0 it follows that $(\operatorname{ad} e)^{2q} \neq 0$. On the other hand, since $e \in \mathfrak{n}$, it follows that for any $\phi \in \Delta$ we can write

$$(\operatorname{ad} e)^{2q}(e_\phi) = \sum_{\xi \in \Delta} b_\xi e_\xi \tag{5.3.6}$$

and, moreover, $b_\xi \neq 0$ implies

$$o(\xi) \geqq o(\phi) + 2q.$$

But this together with (5.3.4) implies that (5.3.6) vanishes for all $\phi \neq -\psi$. Thus $(\operatorname{ad} e)^{2q}(e_{-\psi}) \neq 0$ and in fact, using (5.3.4) once more,

$$(\operatorname{ad} e)^{2q} e_{-\psi} = a' e_\psi$$

for some non-zero scalar a'. Now write $e = e_1 + e_2$, where $e_1 \in \mathfrak{g}_1$ and $e_2 \in \mathfrak{n}_2$.

Expanding $(\operatorname{ad} e)^{2q} = (\operatorname{ad} e_1 + \operatorname{ad} e_2)^{2q}$ it is clear again from (5.3.4) that

$$(\operatorname{ad} e)^{2q}(e_{-\psi}) = (\operatorname{ad} e_1)^{2q}(e_{-\psi}) = a' e_\psi.$$

But for any i, writing

$$(\operatorname{ad} e_1)^i(e_{-\psi}) = \sum_{\xi \in \Delta} b'_\xi e_\xi,$$

it follows that since $a_{\alpha_j} = 0$, one can have $b'_\xi \neq 0$ only when upon writing

$$\xi = \sum_{i=1}^{l} t_i \alpha_i,$$

we have $t_j = -q_j$. In particular, setting $i = 2q$ this means $q_j = -q_j$ or $q_j = 0$. This contradicts (5.3.3) and hence Theorem 5.3 is proved when $\mathfrak{g}$ is simple. The general case follows immediately upon writing $\mathfrak{g}$ as a direct sum of its simple ideals. The component of e in each ideal is necessarily principal nilpotent in that ideal by Corollary 5.3. Q.E.D.

5.4. A theorem proved in [1] (See [1], Remark p. 66.) asserts that every solvable subalgebra $\mathfrak{s}_1$ of $\mathfrak{g}$ is conjugate to a subalgebra $\mathfrak{s}'_1$ of $\mathfrak{s}$. If in addition it is assumed that the elements of $\mathfrak{s}_1$ are all nilpotent (hence $\mathfrak{s}_1$ is a nilpotent Lie algebra by Engel's theorem), it is clear then that $\mathfrak{s}'_1 \subseteq \mathfrak{n}$. In particuar, it follows then that every nilpotent element $e \in \mathfrak{g}$ is conjugate to some element $e' \in \mathfrak{n}$. Actually, we don't need the result of [1] referred to above to prove this. For completeness, we observe that this follows from Lemma 5.1.

LEMMA 5.4. *Any nilpotent element $e \in \mathfrak{g}$ is conjugate to an element $e' \in \mathfrak{n}$.*

Proof. Applying Theorem 3.4 it suffices only to show that the nilpositive element of any S-triple containing x_j (using the notation of Lemma 5.1) as neutral element lies in $\mathfrak{n}$. But this is clear since $x_j \in D$. That is, for $\phi \in \Delta$, $(x_j, \phi) = 1$ implies $\phi \in \Delta^+$.

As a corollary to Theorem 5.3, its proof, and Lemma 5.4 we obtain another characterization of principal nilpotent elements in case $\mathfrak{g}$ is simple.

COROLLARY 5.4. *Assume $\mathfrak{g}$ is simple. Let ψ be the highest root and let $q = o(\psi)$. Let $e \in \mathfrak{g}$ be any nilpotent element. Then e is principal nilpotent if and only if $(\operatorname{ad} e)^{2q} \neq 0$. However, if e is principal nilpotent $(\operatorname{ad} e)^{q+1} = 0$.*

5.5. Corollary 3.7 sets up a natural one-one relation between the conjugate classes of nilpotent elements and the conjugate classes of TDS. It is clear that in this correspondence the class of principal nilpotent elements corresponds to the class of principal TDS. Regarding the latter as distinctive among all conjugate classes of TDS will be given further justification then when it is shown that the former is distinctive among conjugate classes of nilpotent elements. The following corollary shows this very clearly.

COROLLARY 5.5. *The set of principal nilpotent elements (a conjugate class of the adjoint group G) in $\mathfrak{g}$ forms an open, dense and connected subset of the set of all nilpotent elements in $\mathfrak{g}$.*

Proof. Openness follows easily from Corollary 5.3 by choosing a basis

of $\mathfrak{g}$ and considering the $(n-l)\times(n-l)$ minors of the matrix defined by $\operatorname{ad} e$, e nilpotent, with respect to the basis. Denseness follows from Theorem 5.3 and Lemma 5.4. Connectivity is immediate also since the set of principal nilpotent elements is an orbit of the group G.

5.6. Principal nilpotent elements behave like the "regular" elements in the set of all nilpotent elements. That is, one can make a strong case for the following analogy: The set of principal nilpotent elements is to the set of all nilpotent elements as the set of all regular elements is to the set of all semi-simple elements. We cite for example Corollary 5.3 and Corollary 5.5. In this analogy, between the semi-simple elements and the nilpotent elements, the Cartan subalgebra clearly corresponds to the maximum Lie subalgebra of nilpotent elements (see Lemma 5.4 § 2.1. Also all maximum Lie subalgebras of nilpotent elements are conjugate—see § 5.4). One knows that a regular element can be characterized by the property that it lies in one and only one Cartan subalgebra. Corollary 5.6 asserts that also in this regard the analogy still holds.

COROLLARY 5.6. *Let $e\in\mathfrak{g}$ be nilpotent. (One knows that e can be embedded in at least one maximal Lie subalgebra of nilpotent elements.) Then e is principal nilpotent if and only if e lies in one and only one maximal Lie subalgebra of nilpotent elements of $\mathfrak{g}$.*

Proof. By Lemma 5.4 we may assume $e\in\mathfrak{n}$ (using the notation § 5.4). Assume that e is not principal. We will prove e is contained in a second (different) maximal Lie subalgebra of nilpotent elements $\mathfrak{n}'$. Now we may write

$$e=\sum_{\phi\in\Delta^+}a_\phi e_\phi.$$

By Theorem 5.3 since e is not principal, $a_{\alpha_i}=0$ for some value of i. But now it is known (and easy to verify) that

$$\mathfrak{n}'=\sum_{\substack{\phi\in\Delta^+\\ \phi\neq\alpha_i}}(e_\phi)+(e_{-\alpha_i})$$

is a maximal Lie subalgebra of nilpotent elements of $\mathfrak{g}$. (In fact, $\mathfrak{n}$ is carried onto $\mathfrak{n}'$ by any element of G which (1) leaves $\mathfrak{h}$ invariant and (2) whose restriction to $\mathfrak{h}$ is the reflection R_{α_i}—see § 7.1). Obviously $e\in\mathfrak{n}'$.

Now assume e is principal. In fact, we may take $e=e_0$, where we use the notation of § 5.2. Assume $e_0\in\mathfrak{n}'$, where $\mathfrak{n}'=A\mathfrak{n}$, $A\in G$. We shall use

the prime ($'$) on previous notation to indicate the effect of conjugation by A. Applying Theorem 5.3 to $\mathfrak{n}'$ it follows that

$$e_0 = \sum_{\phi \in \Delta^+} a_\phi e_\phi',$$

where $a_\phi \neq 0$ for $\phi = \alpha_i$, $i = 1, 2, \cdots, l$. Now by Theorem 4.3 there exists $A_1 \in N'$ such that

$$A_1 e_0 = \sum_{i=1}^{l} a_{\alpha_i} e_{\alpha_i}'.$$

But then since x_0' ($= Ax$) and $A_1 e_0$ are clearly the neutral and nil-postive elements of an S-triple (see proof of Lemma 5.2), it follows from Theorem 3.6 that

$$A_1 x_0 \in x_0' + \mathfrak{g}^{A_1 e_0}.$$

However, since x_0' is regular upon applying § 2.5(d) and (5.1.2) in the case of this S-triple, it follows that $\mathfrak{g}^{A_1 e_0} \subseteq \mathfrak{n}'$. But then by Theorem 4.3 there exists $A_2 \in N'$ such that $A_2 A_1 x_0 = x_0'$. But now since x_0 is regular, we must have $A_2 A_1 \mathfrak{h} = \mathfrak{h}'$. But this of course implies $\mathfrak{h} \subseteq \mathfrak{s}'$ because $A_1^{-1} A_2^{-1} \in S'$. But this means $\mathfrak{s}'$ and hence $\mathfrak{n}' = [\mathfrak{s}', \mathfrak{s}']$ are stable under $\operatorname{ad} \mathfrak{h}$. But then $\mathfrak{n}'$ must contain and in fact must be spanned by root vectors associated with $\mathfrak{h}$ (since obviously $\mathfrak{n}' \cap \mathfrak{h} = 0$). But $e_0 = \sum_{i=1}^{l} c_i e_{\alpha_i} \in \mathfrak{n}'$ and $c_i \neq 0$. Thus $e_{\alpha_i} \in \mathfrak{n}'$, $i = 1, 2, \cdots, l$. Hence since the e_{α_i} generate $\mathfrak{n}$, it follows that $\mathfrak{n} \subseteq \mathfrak{n}'$. But then of course $\mathfrak{n} = \mathfrak{n}'$. Q. E. D.

5.7. We continue with previous notation.

Now consider $\mathfrak{g}^{e_0}$, the kernel of $\operatorname{ad} e_0$. By Corollary 5.3

$$\dim \mathfrak{g}^{e_0} = l.$$

In the special case when $\mathfrak{g}$ is the set of all $(l+1) \times (l+1)$ complex matrices of trace zero one sees easily that $\mathfrak{g}^{e_0}$ is a commutative Lie algebra (of nilpotent matrices). This and other evidence suggested to us that perhaps $\mathfrak{g}^{e_0}$ is commutative in the general case. However, since among other things we were unable to construct a "useable" basis of $\mathfrak{g}^{e_0}$ we could not settle the question using purely algebraic methods. Nevertheless, it is true that $\mathfrak{g}^{e_0}$ is commutative (Corollary 5.8) in the general case. The proof which we have found is very simple but uses limit arguments. It was Corollary 5.8 which orignally suggested the validity of Theorem 6.7.

Theorem 5.7 or Corollary 5.7 may be regarded as a generalization of the fact that a Cartan subalgebra of a semi-simple Lie algebra is commutative.

THEOREM 5.7. *Let* $\mathfrak{g}$ *be a complex semi-simple Lie algebra of rank* l. *Let* $y \in \mathfrak{g}$ *be arbitrary. Then* $\mathfrak{g}^y$ *contains an* l *dimensional commutative Lie subalgebra.*

Proof. It is well known that the set of regular elements in $\mathfrak{g}$ is dense in $\mathfrak{g}$. Thus we may find a sequence y'_n, $n = 1, 2, \cdots$ of regular elements converging to y.

Now consider the Grassmann manifold of all l planes in $\mathfrak{g}$. This, of course, is compact and hence we may find a subsequence y_n of the sequence y'_n with the property $y_n \to y$ and the Cartan subalgebras $\mathfrak{g}^{y_n}$ converge to an l-plane $\mathfrak{u}$ in the Grassmann manifold. Now if w_i, $i = 1, 2, \cdots, l$, is any basis of $\mathfrak{u}$ we may find elements $w_i^n \in \mathfrak{g}$, $n = 1, 2, \cdots$, $i = 1, 2, \cdots, l$, such that $w_i^n \in \mathfrak{g}^{y_n}$ and $w_i^n \to w_i$ as $n \to \infty$ for $i = 1, 2, \cdots, l$. Since $[y_n, w_i^n] = 0$, it follows immediately by taking the limit that $\mathfrak{u} \subseteq \mathfrak{g}^y$. But $[w_i^n, w_j^n] = 0$. Again taking the limit this obviously implies $\mathfrak{u}$ is commutative. Q. E. D.

COROLLARY 5.7. *Let* $y \in \mathfrak{g}$. *Assume* $\dim \mathfrak{g}^y = 1$. *Then* $\mathfrak{g}^y$ *is commutative.*

5.8. Corollary 5.7 and Corollary 5.3 imply

COROLLARY 5.8. *Let* e *be a principal nilpotent element in* $\mathfrak{g}$. *Then* $\mathfrak{g}^e$ *is commutative.*

6. The principal element of G and the duality theorem.

1. We shall assume from now on that $\mathfrak{g}$ is simple. The theorems to be proved are either true in the general semi-simple case or can be obviously modified to be true in that case. The extension from the simple case to the semi-simple case in any event is immediate. We consider only the former mainly for notational simplicity.

Recall what is meant by a compact form $\mathfrak{k}$ of $\mathfrak{g}$. This may be defined as a real Lie subalgebra of $\mathfrak{g}$ with the property

$$\mathfrak{g} = \mathfrak{k} + i\mathfrak{k} \tag{1}$$

is a real direct sum. (That is, $\mathfrak{k}$ is just a "real form" of $\mathfrak{g}$ and

(2) The restriction of B to $\mathfrak{k}$ is negative definite.

One immediate consequence of (2) is

(a) every element of $\mathfrak{k}$ is semi-simple. Next we recall some facts in the Cartan subalgebra theory of $\mathfrak{k}$ (which is somewhat different from that of $\mathfrak{g}$).

(b) If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$, then $\mathfrak{h}_1 = \mathfrak{t} + i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_1^{\#} = i\mathfrak{t}$.

(c) A Lie subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is a Cartan subalgebra if and only if it is maximal commutative.

An immediate consequence of (c) which we shall soon use is

(d) Any commutative Lie subalgebra $\mathfrak{u}$ of $\mathfrak{k}$ can be imbedded in a Cartan subalgebra of $\mathfrak{k}$.

Corresponding to the adjoint operation (conjugate transpose) in the space of matrices we introduce a *-operation in $\mathfrak{g}$ by defining for any $z \in \mathfrak{g}$

$$z^* = x - iy,$$

where z is written

$$z = x + iy$$

for $x, y \in i\mathfrak{k}$.

Generalizing the notion of a normal matrix or normal operator we will say $z \in \mathfrak{g}$ is normal with respect to $\mathfrak{k}$ whenever

$$[z, z^*] = 0$$

Writing $z = x + iy$, $x, y \in i\mathfrak{k}$ it is clear that z is normal if and only if $[x, y] = 0$.

It is well known of course that a normal matrix is diagonizable (semi-simple). The following generalization of this fact (and also of (a)) is a useful criterion for semi-simplicity in $\mathfrak{g}$.

LEMMA 6.1. *Let z, an element of $\mathfrak{g}$, be normal with respect to a compact form $\mathfrak{k}$ of $\mathfrak{g}$. Then z is a semi-simple element of $\mathfrak{g}$.*

Proof. We may write $z = x + iy$, where ix, $iy \in \mathfrak{k}$. But now since z is normal $[ix, iy] = 0$. Thus by (d) there exists a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ which contains ix and iy. But by (b) $\mathfrak{h}_1 = \mathfrak{t} + i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. But then $z \in \mathfrak{h}_1$ and hence z is semi-simple. Q. E. D.

When the root vectors, relative to a Cartan subalgebra, are suitably normalized (Weyl's normal form) Weyl has given a basis of a compact form of $\mathfrak{g}$ in terms of these root vectors and the Cartan subalgebra. In previous sections we required no other normalization of the e_ϕ, $\phi \in \Delta$, other than (5.1.0). We will assume from here on, unless specified otherwise (in Theorem 8.4), that the root vectors are normalized into Weyl's normal form, with the exception that $(e_\phi, e_{-\phi}) = 1$ is retained in preference to $(e_\phi, e_{-\phi}) = -1$. (In such a normal form one may choose the root vectors e_{α_i},

$i=1,2,\cdots,l$, arbitrarily and hence we need not regard e_0 as having been altered). Then one knows that the linear span, with real coefficients, of $i\mathfrak{h}^\#$ and the vectors $e_\phi - e_{-\phi}$, $ie_\phi + ie_{-\phi}$ for all $\phi \in \Delta^+$ is a compact form $\mathfrak{k}$ of $\mathfrak{g}$. (See remark, p. 11-11 in [12]). One sees easily then that $(e_\phi)^* = e_{-\phi}$ or

$$(6.1.1) \qquad (\sum_{\phi\in\Delta} a_\phi e_\phi)^* = \sum_{\phi\in\Delta} \bar{a}_\phi e_{-\phi}$$

for any set of complex numbers a_ϕ, $\phi \in \Delta$.

6.2. We recall now (see § 6.1) that $\mathfrak{g}$ is simple. Let $\psi \in \Delta$ be the highest root.[8] Let $\Pi^Q \subseteq \Delta$ be the subset of $l+1$ roots obtained by adjoining $-\psi$ to the simple positive roots. That is, $\Pi^Q = \Pi \cup (-\psi)$. The notion of simple root and highest root are notions which of course are relative to the choice of a lexicographical ordering in $\mathfrak{h}^\#$ or rather to the choice of a Weyl chamber (see § 5.1). We will say then that the roots in Π^Q are Q-simple relative to the chamber D. (We shall not require the fact but it can be shown that the number of Weyl chambers which give rise to the same set of Q-simple roots is equal to order of the fundamental group of G; that is, to the order of the center of the simply connected covering group of G.)

Now an element $z \in \mathfrak{g}$ will be called cyclic if there exists a Cartan subalgebra $\mathfrak{h}_1$ and a set $\Pi_1{}^Q \subseteq \Delta_1$ of Q-simple roots relative to some Weyl chamber D_1 in $\mathfrak{h}_1{}^\#$ such that z can be written

$$(6.2.1) \qquad z = \sum_{\xi\in\Pi_1{}^Q} a_\xi e_\xi,$$

where the a_ξ, $\xi \in \Pi_1{}^Q$, are non-zero complex numbers and the e_ξ are root vectors for $\mathfrak{h}_1$ corresponding to the roots $\xi \in \Delta_1$.[9] Cyclic elements play a major role in the remainder of this paper.

If z is given by (6.2.1), observe that in effect we have formed the cyclic element z by adding $a_{-\psi_1}e_{-\psi_1}$ to a principal nilpotent element, where ψ_1 is the highest root in Δ_1 relative to D_1. It shall be shown that not only does this destroy nilpotency but the cyclic elements are in fact regular. First we shall need

LEMMA 6.2. *Let the cyclic elements z, $z' \in \mathfrak{g}$ be given by*

$$z = \sum_{\beta\in\Pi^Q} a_\beta e_\beta$$

and

$$z' = \sum_{\beta\in\Pi^Q} a'_\beta e_\beta,$$

[8] The properties of ψ which will be required are all consequences of (5.3.5).

[9] The set Δ_1 is the set of roots associated with the Cartan subalgebra $\mathfrak{h}_1$.

where the coefficients a_β, a'_β are all arbitrary non-zero complex numbers. Then if H is the subgroup of G corresponding to $\mathfrak{h}$ there exists an element $A \in H$ and a non-zero scalar λ such that

$$Az = \lambda z'.$$

Proof. Let ϵ_i, $i = 1, 2, \cdots, l$, be as in § 5.1. Let $y \in \mathfrak{h}$ be given by

$$y = \sum_{i=1}^{l} \operatorname{Log}(a'_{\alpha_i}/a_{\alpha_i})\epsilon_i.$$

Clearly $\operatorname{Exp} y \in H$ and

$$\operatorname{Exp} y\Big(\sum_{i=1}^{l} a_{\alpha_i} e_{\alpha_i}\Big) = \sum_{i=1}^{l} a'_{\alpha_i} e_{\alpha_i}.$$

Now let c be any complex number and let x_0 be given by (5.2.1). Then clearly

$$\operatorname{Exp}(y + cx_0)\Big(\sum_{i=1}^{l} a_{\alpha_i} e_{\alpha_i}\Big) = e^c\Big(\sum_{i=1}^{l} a'_{\alpha_i} e_{\alpha_i}\Big). \tag{6.2.1}$$

Now let $q = o(\psi)$ as in § 5.3. But then

$$\operatorname{Exp} cx_0(e_{-\psi}) = e^{-cq} e_{-\psi}$$

and if b is defined by $\operatorname{Exp} y(e_{-\psi}) = be_{-\psi}$, then $b \neq 0$ and

$$\operatorname{Exp}(y + cx_0)(a_{-\psi} e_{-\psi}) = e^{-cq} b a_{-\psi} e_{-\psi}. \tag{6.2.2}$$

Now choose c so that

$$e^{-c(q+1)} = a'_{-\psi}/ba_{-\psi}.$$

Then

$$e^{-cq} b a_{-\psi} = e^c a'_{-\psi}. \tag{6.2.3}$$

Hence if $\lambda = e^c$, $A = \operatorname{Exp}(y + cx_0)$, one has $A \in H$ and (6.2.1), (6.2.2), and (6.2.3) imply

$$A\Big(\sum_{\beta \in \Pi^q} a_\beta e_\beta\Big) = \lambda\Big(\sum_{\beta \in \Pi^q} a'_\beta e_\beta\Big). \qquad \text{Q. E. D.}$$

An immediate consequence of Lemma 6.2 is the following theorem which asserts that up to scalar multiplication the set of cyclic elements forms a single conjugate class in $\mathfrak{g}$.

THEOREM 6.2. *Let z and z' be any two cyclic elements in $\mathfrak{g}$. Then there exists a scalar λ, $\lambda \neq 0$, such that z and $\lambda z'$ are conjugate to each other.*

Proof. If D_1 is a Weyl chamber in $\mathfrak{h}_1{}^{\#}$, where $\mathfrak{h}_1$ is a Cartan subalgebra, there exists $A \in G$ such that $A\mathfrak{h}_1 = \mathfrak{h}_1$ and $AD_1 = D$. This fact together with Lemma 6.2 proves Theorem 6.2. Q. E. D.

6.3. Lemma 6.1 is needed solely to prove the following lemma.

LEMMA 6.3. *Cyclic elements are semi-simple.*

Proof. Now if we write

$$\psi = \sum_{i=1}^{l} q_i \alpha_i, \tag{6.3.1}$$

recall that the coefficients q_i are positive integers. (See (5.3.3)). Thus if e_1 is defined by

$$e_1 = \sum_{i=1}^{l} (q_i)^{\frac{1}{2}} e_{\alpha_i}$$

and z_1 is defined by

$$z_1 = e_1 + e_{-\psi},$$

then e_1 and z_1 are, respectively, principal nilpotent and cyclic elements.

To prove Lemma 6.3 it suffices by Theorem 6.2 to prove only that z_1 is semi-simple. But to prove that z_1 is semi-simple, by Lemma 6.1 it suffices only to prove that z_1 is normal with respect to $\mathfrak{k}$. Now by (6.1.1)

$$z_1^* = e_1^* + e_\psi$$

and

$$e_1^* = \sum_{i=1}^{l} (q_i)^{\frac{1}{2}} e_{-\alpha_i}.$$

But, obviously then, since ψ is the highest root

$$[e_1, e_\psi] = [e_1^*, e_{-\psi}] = 0.$$

Thus

$$[z_1, z_1^*] = [e_1 + e_{-\psi}, e_1^* + e_\psi] = [e_1, e_1^*] + [e_{-\psi}, e_\psi].$$

But

$$[e_1, e_1^*] = \sum_{i=1}^{l} q_i [e_{\alpha_i}, e_{-\alpha_i}] = \sum_{i=1}^{l} q_i \alpha_i = \psi$$

by (5.1.1) and (6.3.1). On the other hand, of course,

$$[e_{-\psi}, e_\psi] = -\psi.$$

Thus $[z_1, z_1^*] = 0$ and hence z_1 is normal with respect to $\mathfrak{k}$. Q. E. D.

6.4. To gain information on cyclic elements in general it suffices by Theorem 6.2 to focus attention on a single fixed cyclic element. We choose this element to be z_0, where z_0 is given by

$$z_0 = e_0 + e_{-\psi}. \tag{6.4.1}$$

Here, of course, e_0 is given by (5.2.2).

As in § 5.2 let $\mathfrak{a}_0$ be the TDS spanned by x_0, e_0 and f_0. Now let n_k be the multiplicity of the irreducible $2k+1$-dimensional representation π_{2k+1} in the complete reduction of the adjoint representation of $\mathfrak{a}_0$ on $\mathfrak{g}_0$. By Corollary 5.2, $n_k = 0$ if k is a half-integer and

$$l = n_0 + n_1 + n_2 + \cdots. \tag{6.4.2}$$

Now it follows from § 2.5(d) and (5.2.4) that the kernel $\mathfrak{g}^{e_0}$ of $\operatorname{ad} e_0$ is contained in $\mathfrak{s}$. On the other hand, it is obvious that $\mathfrak{g}^{e_0} \cap \mathfrak{h} = 0$. Therefore, since $x_0 \in \mathfrak{h}$ is regular, it follows that the eigenvalues of $\operatorname{ad} x_0$ on $\mathfrak{g}^{e_0}$ are positive. This implies two things, (1) $\mathfrak{g}^{e_0} \subseteq \mathfrak{n}$ (Actually, we have already noted this fact—see proof of Corollary 5.6) and (2) by § 2.5(d), $n_0 = 0$. That is, only the zero element is annihilated by $\operatorname{ad} \mathfrak{a}_0$. Indeed, this also follows from § 2.5(h) which asserts

$$\begin{aligned} n_0 &= \dim \mathfrak{g}_0 - \dim \mathfrak{g}_1 \\ &= l - l \\ &= 0. \end{aligned}$$

For the present we direct our attention to the first fact, $\mathfrak{g}^{e_0} \subseteq \mathfrak{n}$.

Let $\mathfrak{n}^*$ be the maximal Lie subalgebra of nilpotent elements given by

$$\mathfrak{n}^* = \sum_{\phi \in \Delta^+} (e_{-\phi}).$$

Then, of course,

$$\mathfrak{g} = \mathfrak{n}^* + \mathfrak{h} + \mathfrak{n} \tag{6.4.3}$$

is a direct sum decomposition. We shall let ρ (resp. ρ^*) be the projection

$$\rho : \mathfrak{g} \to \mathfrak{n}$$
$$(\text{resp. } \rho^* : \mathfrak{g} \to \mathfrak{n}^*)$$

of $\mathfrak{g}$ onto $\mathfrak{n}$ (resp. $\mathfrak{n}^*$) defined by the decomposition (6.4.3).

Now consider the kernel $\mathfrak{g}^{z_0}$ of $\operatorname{ad} z_0$. A relation between $\mathfrak{g}^{e_0}$ and $\mathfrak{g}^{z_0}$ is given by

LEMMA 6.4A. *Let ρ_0 be the restriction of the projection to the subspace $\mathfrak{g}^{z_0}$. Then $\rho_0(\mathfrak{g}^{z_0}) \subseteq \mathfrak{g}^{e_0}$ and in fact*

$$\rho_0 : \mathfrak{g}^{z_0} \to \mathfrak{g}^{e_0}$$

is a linear isomorphism of $\mathfrak{g}^{z_0}$ onto $\mathfrak{g}^{e_0}$.

Proof. It follows immediately from the decomposition (6.4.3) of $\mathfrak{g}$ and (5.1.3) that

$$[e_{-\psi}, \mathfrak{n}] \subseteq \mathfrak{n}^* + \mathfrak{h}$$
$$[e_{-\psi}, \mathfrak{h}] \subseteq (e_{-\psi})$$
$$[e_{-\psi}, \mathfrak{n}^*] = 0.$$

Now let $y \in \mathfrak{g}^{z_0}$ be arbitrary. Write

$$y = v + x + u,$$

where $v \in \mathfrak{n}^*$, $x \in \mathfrak{h}$, and $u \in \mathfrak{n}$.

Now by (6.4.1) and § 5.1

$$\begin{aligned} 0 &= [z_0, y] \\ &= [e_0, v] + [e_0, x] + [e_0, u] \\ &\quad + (x, \psi) e_{-\psi} + [e_{-\psi}, u]. \end{aligned} \tag{6.4.4}$$

Applying the projection ρ to both sides of (6.4.4) we obtain

$$[e_0, x] + [e_0, u] = 0.$$

But then recalling the relation $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, it is clear that $[e_0, x] = 0$ and $[e_0, u] = 0$. Thus since $u = \rho(y)$ it follows that $\rho(\mathfrak{g}^{z_0}) \subseteq \mathfrak{g}^{e_0}$. It also follows that since $\mathfrak{g}^{e_0} \subseteq \mathfrak{n}$, we must have

$$x = 0. \tag{6.4.5}$$

Thus $y = v + u$ and

$$0 = [e_0, v] + [e_0, u] + [e_{-\psi}, u].$$

To show first that ρ_0 is one-one, assume $u = 0$. But this and the last expression imply that $v \in \mathfrak{g}^{e_0}$. But $\mathfrak{g}^{e_0} \subseteq \mathfrak{n}$. Hence $v = 0$. Thus $y = 0$ which proves that ρ_0 is an isomorphism into. But now, by Corollary 5.3, $\dim \mathfrak{g}^{e_0} = l$. On the other hand, $\dim \mathfrak{g}^{z_0} \geqq l$ (see, for example Theorem 5.7). Since ρ_0 is an isomorphism, this means

$$\dim \mathfrak{g}^{z_0} = l$$

and also that ρ_0 is onto. Q. E. D.

A major consequence of Lemma 6.4A is

COROLLARY 6.4. *Cyclic elements are regular.*

Proof. As above let z_0 be defined by (6.4.1). It suffices to show z_0 is regular. But by Lemma 6.3 z_0 is semi-simple. Hence $\mathfrak{g}^{z_0}$ contains a Cartan subalgebra. But, by Lemma 6.4, $\dim \mathfrak{g}^{z_0} = l$. Hence z_0 is regular.

Corollary 6.4 implies

LEMMA 6.4B. *Let z_0 be defined by* (6.4.1). *Then $\mathfrak{g}^{z_0}$ is a Cartan subalgebra.*

We will let $\mathfrak{h}'$ designate the Cartan subalgebra $\mathfrak{g}^{z_0}$.

In the proof of Lemma 6.4 we considered the decomposition of any element $y \in \mathfrak{h}'$ and showed that its projection x in $\mathfrak{h}$ vanishes. (See (6.4.5)). We state this as

LEMMA 6.4C. *Let z_0 be defined by* (6.4.1). *Let ρ and ρ^* be the projections of $\mathfrak{g}$ on $\mathfrak{n}$ and $\mathfrak{n}^*$ also defined as in* § 6.4. *Then the projection $1-(\rho+\rho^*)$ of $\mathfrak{g}$ on $\mathfrak{h}$ vanishes on $\mathfrak{h}'$.*

Perhaps a simpler way of expressing Lemma 6.4C is to state that the Cartan subalgebras $\mathfrak{h}'$ and $\mathfrak{h}$ are orthogonal to each other (with respect to B).

6.5. Now $\mathfrak{g}^{e_0}$ is invariant under $\operatorname{ad} x_0$ (see § 2.5(d)). Let u_i, $i=1,2,\cdots,l$, be a basis of $\mathfrak{g}^{e_0}$ and also eigenvectors of $\operatorname{ad} x_0$. Let k_i, $i=1,2,\cdots,l$, be the corresponding eigenvalues. That is, $u_i \in \mathfrak{g}_{k_i}$, $i=1,2,\cdots,l$. We may regard the basis so ordered that $k_i \leqq k_{i+1}$. At this stage we already know two of the k_i, namely, the extreme ones k_1 and k_l. We also know an inequality, $k_{l-1} < k_l$.

LEMMA 6.5A. *Let k_i be defined as above. Then, where $q = o(\psi)$,*

$$1 = k_1 \leqq k_i \leqq \cdots \leqq k_{l-1} < k_l = q.$$

Proof. As we noted in § 6.4 we must have $k_1 > 0$. On the other hand, since $e_0 \in \mathfrak{g}^{e_0}$, it follows that $k_1 = 1$. But also $e_\psi \in \mathfrak{g}^{e_0}$ since ψ is the highest root. Thus $[x_0, e_\psi] = qe_\psi$ implies $k = q$. Since $o(\phi) < q$ for every $\phi \neq \psi$ (see (5.3.4)), it also follows that $k_{l-1} < k_l$. Q.E.D.

Taking the proof of Lemma 6.5A into account we will choose $u_1 = e_0$ and $u_l = e_\psi$.

Now let d_i, $i=1,2,\cdots,l$, be the dimensions, in non-decreasing order, of the irreducible components occurring in the complete reduction of adjoint representation of a principal TDS (e.g. $\mathfrak{a}_0$) on $\mathfrak{g}$. Appplying § 2.5(d) the eigenvalues k_i yield the dimensions d_i by the relation

$$d_i = 2k_i + 1. \tag{6.5.1}$$

Lemma 6.5A implies $d_1 = 3$, $d_l = 2q+1$ and $d_{l-1} < 2q+1$.

Now by Lemma 6.4A there exists a unique basis $y_1, y_2, \cdots, y_l$ of the Cartan subalgebra $\mathfrak{h}'$ with the property that $\rho(y_i) = u_i$. Define v_i by

$$y_i = u_i + v_i. \tag{6.5.2}$$

By Lemma 6.4C it follows that $v_i \in \mathfrak{n}^*$ for $i = 1, 2, \cdots, l$. Since $u_1 = e_0$ and since $\rho(z_0) = e_0$, it follows from (6.4.1) and Lemma 6.4A that $v_1 = e_{-\psi}$ and $y_1 = z_0$. Now $u_l = e_\psi$. We shall have to know what v_l is.

LEMMA 6.5B. *Let c_i and q_i, $i = 1, 2, \cdots, l$, be defined respectively by* (5.2.2) *and* (6.3.1). *Then v_l is the element given by*

$$v_l = \sum_{i=1}^{l} (q_i/c_i) e_{-\alpha_i}. \tag{6.5.3}$$

Proof. As in the proof of Lemma 6.3

$$\begin{aligned}[z_0, e_\psi + \sum_{i=1}^{l} (q_i/c_i) e_{-\alpha_i}] &= [e_{-\psi}, e_\psi] + \sum_{i=1}^{l} q_i [e_{\alpha_i}, e_{-\alpha_i}] \\ &= -\psi + \sum_{i=1}^{l} q_i \alpha_i = 0.\end{aligned}$$

Thus $e_\psi + \sum_{i=1}^{l} (q_i/c_i) e_{-\alpha_i} \in \mathfrak{h}'$. It follows immediately then from Lemma 6.4A that $v_l = \sum_{i=1}^{l} (q_i/c_i) e_{-\alpha_i}$. Q. E. D.

6.6. Now it is clear (for example from the matrix representation of a TDS given in § 2.5) that there exists an element A in the subgroup of G corresponding to $\mathfrak{a}_0$ such that $Ax_0 = -x_0$. But since x_0 is regular, it is clear then that

$$A : \mathfrak{h} \to \mathfrak{h},$$
$$A : \mathfrak{n}^* \to \mathfrak{n}.$$

In fact these relations are already contained in the more general fact

$$A : \mathfrak{g}_j \to \mathfrak{g}_{-j}$$

for any $-q \leqq j \leqq q$. It follows then that we may interchange the roles of Δ^+ and $-\Delta^+$, $\mathfrak{n}$ and $\mathfrak{n}^*$ and also $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ in the results of §§ 6.2-5. But then one sees that v_l is principal nilpotent and that $y_l = u_l + v_l$ is cyclic. But then by Corollary 6.4 it follows that $\mathfrak{g}^{y_l} = \mathfrak{h}'$. Finally, applying Lemma 6.4A we obtain

LEMMA 6.6. *We retain previous notation. Let ρ^*_0 be the restriction of ρ^* to $\mathfrak{h}'$. Then* (1) *v_l is a principal nilpotent element,* (2), *$\rho^*_0(\mathfrak{h}') \subseteq \mathfrak{g}^{v_l}$ and in fact*

$$\rho^*_0 : \mathfrak{h}' \to \mathfrak{g}^{v_l}$$

is isomorphism onto. In other words, the elements v_i, $i = 1, 2, \cdots, l$, *form a basis of* $\mathfrak{g}^{v_1}$. *Also, the elements* v_i *commute with each other.* (*See Corollary* 5.8).

It is obvious from (5.2.3) that v_l is a nil-negative element of a principal S-triple containing x_0 as neutral element or an nil-positive element of a principal S-triple containing $-x_0$ as neutral element. Thus $\mathfrak{g}^{v_1}$ is stable under $\operatorname{ad} x_0$ and since all principal S-triples are conjugate the eigenvalues of $\operatorname{ad} x_0$ on $\mathfrak{g}^{v_1}$ are $-k_l, -k_{l-1}, \cdots, -k_1$ in non-decreasing order. The elements v_i are a basis of $\mathfrak{g}^{v_1}$ by Lemma 6.6 but it is not at all clear yet that they are eigenvectors of $\operatorname{ad} x_0$.

6.7. We now isolate a particular conjugate class in G, elements of which, will play a major role in the remainder of this paper. An element $P \in G$ will be called a principal element of G if there exists a principal regular element $x \in \mathfrak{g}$ (see § 5.2) such that P can be written

$$P = \operatorname{Exp}(2\pi i/s)x, \tag{6.7.1}$$

where $s = q + 1$ and q, as usual, is the order of the highest root ψ. Note that since x lies in a principal TDS, the principal element P lies in a subgroup of G corresponding to a principal TDS. (We shall make no use of the fact here but it can be shown that the number of principal regular elements x which satisfy (6.7.1) for a fixed principal element $P \in G$ is equal to order of the fundamental group of G). Various characterizations of principal elements will be given in §§ 8 and 9.

Throughout the remainder of the paper we will let ω be the primitive s root of unity defined by $\omega = e^{2\pi i/s}$. Let P_0 be principle element in G defined by letting $P_0 = \exp(2\pi i/s)x_0$. It is clear then that

$$P_0 u = \omega^j u$$

for $u \in \mathfrak{g}_j$, $-q \leqq j \leqq q$. It follows then that ω^j, $j = 0, 1, \cdots, q$, are the eigenvalues of P_0 and if $\mathfrak{u}_j$ designates the corresponding eigenspaces, then

$$\mathfrak{u}_j = \mathfrak{g}_j + \mathfrak{g}_{j-s}, \tag{6.7.2}$$

where it is understood that $\mathfrak{g}_{-s}$ denotes the zero subspace.

It is an immediate consequence of (6.7.2) that z_0 is an eigenvector of P_0. In fact,

$$P_0 z_0 = \omega z_0. \tag{6.7.3}$$

But this clearly means that the Cartan subalgebra $\mathfrak{h}' = \mathfrak{g}^{z_0}$ is stable under P_0.

On the other hand, the elements $y_i = u_i + v_i$, where $u_i \in \mathfrak{g}_{k_i}$, form a basis of $\mathfrak{h}'$. Let us apply P_0 to these basal elements. Then by § 6.5

$$P_0 y_i = \omega^{k_i} u_i + P_0 v_i.$$

But the elements $P_0 y_i$ belong to $\mathfrak{h}'$ and furthermore since, obviously, $P_0 v_i \in \mathfrak{n}^*$, it follows that

$$\rho_0(P_0 y_i) = \omega^{k_i} u_i.$$

But now if we apply Lemma 6.4A we must have

$$P_0 y_i = \omega^{k_i} y_i.$$

Thus y_i, u_i and hence also v_i are contained in $\mathfrak{u}_{k_i}$. But since $v_i \in \mathfrak{n}^*$, it follows from (6.7.2) that $v_i \in \mathfrak{g}_{k_i - s}$. Thus not only did we prove that the v_i are eigenvectors of $\operatorname{ad} x_0$ but more we obtain a duality relation among the integers k_i.

LEMMA 6.7. *Let v_i, $i = 1, 2, \cdots, l$, be defined by* (6.5.2). *Then $v_i \in \mathfrak{g}_{k_i - s}$, $i = 1, 2, \cdots, l$. Also the integers k_i satisfy the following duality law,*

$$s = k_1 + k_l = k_2 + k_{l-1} = \cdots = k_l + k_1.$$

Proof. Only the second part of Lemma 6.7 is not yet proved. By Lemma 6.6 the elements v_i are a basis of $\mathfrak{g}^{v_1}$ and the first part of Lemma 6.7 asserts that v_i is an eigenvector of $\operatorname{ad} x_0$ with eigenvalue $k_i - s$. On the other hand, as we have seen, the eigenvalues of $\operatorname{ad} x_0$ on $\mathfrak{g}^{v_1}$ are, in non-decreasing order,

$$-k_l \leqq -k_{l-1} \leqq \cdots \leqq -k_1.$$

But these must be identical, and in the same order, as

$$k_1 - s \leqq k_2 - s \leqq \cdots \leqq k_l - s.$$

Subtracting termwise we obtain for all defined p, $k_p + k_{l+1-p} = s$. Q. E. D.

Observe that the duality together with the inequality $k_{l-1} < k_l$ implies the inequality $k_1 < k_2$.

In [3] Chevalley observed empirically a similar duality in the exponents (see [3], p. 24) of $\mathfrak{g}$. Later it will be proved that the integers k_i are in fact these exponents and that the duality observed in the latter is the same as the duality just proved in the former. More than just a numerical coincidence the duality in the k_i takes the following form. (Theorem 6.7 summarizes results which have already been proved.)

THEOREM 6.7. *Let c_i, $i = 1, 2, \cdots, l$, be l arbitrary non-zero complex*

numbers. Let q_i, $i=1,2,\cdots,l$, *be the coefficients of the highest root* ψ *relative to the simple positive roots* α_i.

Let

$$e_0=\sum_{i=1}^{l} c_i e_{\alpha_i}$$

and

$$\tilde{e}_0=\sum_{i=1}^{l} (q_i/c_i)\, e_{-\alpha_i}.$$

Then e_0 *and* $\tilde{e}_0$ *are principal nilpotent elements in* $\mathfrak{g}$ *so that* $\mathfrak{g}^{e_0}$ *and* $\mathfrak{g}^{\tilde{e}_0}$ *are* l*-dimensional and commutative.*

Let $x_0=\sum_{i=1}^{l} \epsilon_i$, *where* ϵ_i, $i=1,2,\cdots,l$, *is the basis of the Cartan subalgebra* $\mathfrak{h}$ *dual to the simple positive roots* α_i, $i=1,2,\cdots,l$, *so that* $(x_0,\phi)=o(\phi)$ *is the order of a root* ϕ. *For any non-zero integer* j *let*

$$\mathfrak{g}_j=\sum_{o(\phi)=j} (e_\phi)$$

be the eigenspace of $\operatorname{ad} x_0$ *for the eigenvalue* j.

Then $\mathfrak{g}^{e_0}$ *and* $\mathfrak{g}^{\tilde{e}_0}$ *are each stable under* $\operatorname{ad} x_0$ *and the eigenvalues of the restriction of* $\operatorname{ad} x_0$ *to* $\mathfrak{g}^{e_0}$ *and* $\mathfrak{g}^{\tilde{e}_0}$, *respectively, are positive integers* k_i *and the negative integers* $-k_i$, $i=1,2,\cdots,l$, *where*

$$1=k_1<k_2\leqq\cdots\leqq k_{l-1}<k_l=q,$$

and $q=\sum_{i=1}^{l} q_i=o(\psi)$ *and* $2k_1+1, 2k_2+1,\cdots,2k_l+1$ *are the dimensions of the irreducible components occuring in the complete reduction of the representation of a principal* TDS *on* $\mathfrak{g}$.

Let

$$z_0=e_0+e_{-\psi}$$

and

$$\tilde{z}_0=\tilde{e}_0+e_\psi$$

(*cycle elements*). *Then* z_0 *and* $\tilde{z}_0$ *are regular, so that* $\mathfrak{g}^{z_0}$ *and* $\mathfrak{g}^{\tilde{z}_0}$ *are Cartan subalgebras; and* $[z_0,\tilde{z}_0]=0$ *so that* $\mathfrak{g}^{z_0}=\mathfrak{g}^{\tilde{z}_0}$.

Let P_0, *a principal element of* G, *be given by* $P_0=\operatorname{Exp}(2\pi i/s)x_0$, *where* $s=q+1$. *Write* $\mathfrak{h}'$ *for* $\mathfrak{g}^{z_0}=\mathfrak{g}^{\tilde{z}_0}$ *Then* $\mathfrak{h}'$ *is orthogonal to* $\mathfrak{h}$. *But much more than this,* $\mathfrak{h}'$ *is stable under* P_0 *and the eigenvalues of* $P_0 \mid \mathfrak{h}'$ *are* ω^{k_i}, $i=1,2,\cdots,l$, *where* $\omega=e^{2\pi i/s}$. *Furthermore, we can find corresponding eigenvectors,* y_i, *which form a basis of* $\mathfrak{h}'$ *and are such that*

$$y_i=u_i+v_i,$$

where the u_i form a basis of $\mathfrak{g}^{e_0}$ and the v_i form a basis of $\mathfrak{g}^{\bar{e}_0}$, $i = 1, 2, \cdots, l$. Moreover, $u_i \in \mathfrak{g}_{k_i}$ and $v_i \in \mathfrak{g}_{k_i - s}$ so that the integers k_i satisfy the duality relation

$$s = k_1 + k_l = k_2 + k_{l-1} = \cdots = k_l + k_1.$$

Finally we can choose $y_1 = z_0$ and $y_l = \bar{z}_0$.

6.8. Among other things, Theorem 6.7 asserts that the integers k_i may be found by considering the eigenvalues, ω^{k_i}, of the restriction of the principal element P_0 to $\mathfrak{h}'$. Theorem 6.8 below asserts that without knowledge of $\mathfrak{h}'$ or the restriction $P_0 \mid \mathfrak{h}'$ we can determine the integers k_i directly by considering multiplicities of eigenvalues of a principal element with respect to its action on $\mathfrak{g}$. In the notation of § 6.4 and Theorem 6.8 this means we can express n_k in terms of s_k.

THEOREM 6.8. *Let P be a principal element of G. Let s_k, $k = 0, 1, \cdots, q$, be the multiplicity of the eigenvalue ω^k of P (In the notation of § 6.7, $s_k = \dim \mathfrak{g}_k + \mathfrak{g}_{k-s}$) and as in § 6.4 let n_k be the multiplicity of π_{2k+1} in the adjoint representation a principal* TDS *on $\mathfrak{g}$. Then*

$$n_k = s_k - l \tag{6.8.1}$$

In other words, let $\mathfrak{h}_1$ be a Cartan subalgebra which is stable under P and is such that the eigenvalues of $P \mid \mathfrak{h}_1$ are ω^{k_i}, $i = 1, 2, \cdots, l$ (such a Cartan subalgebra exists by Theorem 6.7). Then if $\mathfrak{m}_1$ is the B-orthocomplement to $\mathfrak{h}_1$, $\mathfrak{m}_1$ is stable under P and the eigenvalues of $P \mid \mathfrak{m}_1$ are ω^k, $k = 0, 1, \cdots, q$. Furthermore each eigenvalue of $P \mid \mathfrak{m}_1$ occurs with multiplicity l.

Proof. Since any element in G is orthogonal with respect to B, it follows that $\mathfrak{m}_1$ is stable under P. It suffices then to assume $P = P_0$ and $\mathfrak{h}_1 = \mathfrak{h}$ to prove Theorem 6.8 for this case. Now by § 2.5(h)

$$\dim \mathfrak{g}_j - \dim \mathfrak{g}_{j+1} = n_j. \tag{6.8.2}$$

Since $n_0 = 0$ upon summing (6.8.2) we obtain for $0 \leqq k \leqq q$

$$\dim \mathfrak{g}_0 - \dim \mathfrak{g}_k = \sum_{j=1}^{k-1} n_j. \tag{6.8.3}$$

But by § 2.5(f)

$$\dim \mathfrak{g}_{s-k} = \sum_{j=s-k}^{q} n_j. \tag{6.8.4}$$

But by the duality in Theorem 6.7 $n_{s-j} = n_j$. But then subtracting (6.8.3) from (6.8.4) and recalling that $\dim \mathfrak{g}_{s-k} = \dim \mathfrak{g}_{k-s}$ and $\dim \mathfrak{g}_0 = l$ it follows that

$$s_k - l = \dim (\mathfrak{g}_k + \mathfrak{g}_{k-s}) - l = n_k. \qquad \text{Q. E. D.}$$

COROLLARY 6.8. *Let r be the number of positive roots of $\mathfrak{g}$. Let $s = q + 1$, where q is the order of the highest root ψ. Then*

$$ls = 2r.$$

Proof. This immediately follows from the second part of Theorem 6.8 since $\dim m_1 = 2r$. Q. E. D.

Since $n = l + 2r$ note that it also proved that l divides n. In fact $n = l(s+1)$.

7. The notion of the apposition of two Cartan subalgebras.

1. Let $\mathfrak{h}_1$ be any Cartan subalgebra of $\mathfrak{g}$. Let H_1 be the corresponding subgroup of G. Let $N(H_1)$ be the normalizer of H_1 in G. It is obvious that $A \in N(H_1)$ if and only if $\mathfrak{h}_1$ is stable under A. Let W_1 be the subgroup of endomorphisms of $\mathfrak{h}_1$ induced by restricting the elements of $N(H_1)$ to $\mathfrak{h}_1$. Then the restriction defines a homomorphism

$$\zeta: N(H_1) \to W_1$$

of $N(H_1)$ onto W_1. The group W_1 is finite and is called the Weyl group with respect to $\mathfrak{h}_1$. It is well known that the kernel of ζ is H_1 so that $N(H_1)/H_1 \cong W_1$. If $\sigma \in W_1$ and $A \in N(H_1)$ and $A \in N(H_1)$ is such that $\zeta(A) = \sigma$, then A will be called an extension of σ. It is clear then that the most general extension of σ is of the form $A_1 A$, where $A_1 \in H_1$.

Let Δ_1 be the set of roots with respect to $\mathfrak{h}_1$. It is well known that Δ_1 is stable under W_1. For each root $\phi \in \Delta_1$ let R_ϕ be the "reflection" of $\mathfrak{h}_1$ defined by

$$R_\phi y = y - (2(\phi, y)/(\phi, \phi))\phi \tag{7.1.1}$$

One knows that $R_\phi \in W_1$ for all $\phi \in \Delta_1$ and that W_1 is generated by these reflections (See [12], Theorème 1, p. 16-05).

LEMMA 7.1. *Let $\sigma \in W_1$. Let μ be an eigenvalue of σ. Then μ is a primitive j-th root of unity for some j and every other primitive j-th root of unity occurs as an eigenvalue of σ.*

Proof. Since Δ_1 is stable under σ, there exists a basis of $\mathfrak{h}_1$ (e. g. a set of simple positive roots) with respect to which σ is represented by a matrix with rational coefficients. It follows then that the characteristic polynomial of σ also has rational coefficients. Now since σ has finite order, μ must be a primitive j-th root of unity for some integer j. It folows then that the characteristic polynomial of σ is a product of cyclotomic polynomials and

hence every primitive j-th root of unity is a root of the characteristic polynomial. Q. E. D.

7.2. Let $\mathfrak{u}_1$ and Π^Q, the set of Q-simple roots with respect to D, be defined, respectively, as in § 6.7 and § 6.2. It is clear that $\mathfrak{u}_1$ is the set of all elements $y \in \mathfrak{g}$ of the form

$$y = \sum_{\beta \in \Pi^Q} a_\beta e_\beta$$

We know that when all the coefficients a_β are different from zero, y is cyclic, and hence also regular. It is interesting that in case one of the coefficients is zero, not only does y cease to be regular, it is in fact nilpotent.

LEMMA 7.2. *Let Π^Q be as in § 6.2. Let $y \in \mathfrak{g}$ be of the form*

$$y = \sum_{\beta \in \Pi^Q} a_\beta e_\beta.$$

That is, in the notation of § 6.7, assume y satisfies $P_0 y = \omega y$. Then if one of the coefficients a_β is zero, y is nilpotent.

Proof. Assume one of the coefficients is zero. In fact, let β_i, $i = 1, 2, \cdots, l+1$, be the elements of Π^Q. Without loss assume $a_{\beta_{l+1}} = 0$.

Now upon writing $\psi = \sum_{i=1}^{l} q_i \alpha_i$ we know that the coefficients q_i are all non-zero (See (5.3.3)). It follows immediately then that the roots $\beta_1, \beta_2, \cdots, \beta_l$ form a basis of $\mathfrak{h}$. But if we use this basis to define a lexicographical ordering in $\mathfrak{h}^\#$ and set $\bar{\Delta}^+$ equal to the set of positive roots relative to this ordering, it follows that $\beta_i \in \bar{\Delta}^+$ for $i = 1, 2, \cdots, l$. But then $\bar{\mathfrak{n}} = \sum_{\phi \in \bar{\Delta}^+} (e_\phi)$ is a Lie algebra of nilpotent elements and $y \in \bar{\mathfrak{n}}$. Hence y is nilpotent. Q. E. D.

7.3. Applying Lemma 6.2 we obtain the corollary

LEMMA 7.3. *Let $y \in \mathfrak{u}_1$. Let H be the subgroup of G corresponding to $\mathfrak{h}$. Then y is either nilpotent or else there exists $A \in H$ and a non-zero scalar λ such that*

$$y = \lambda A z_0.$$

For any element $A \in G$ let $\mathfrak{g}^A$ be the set of fixed vectors of A. An element $A \in G$ is then said to be regular if (1) A is semi-simple (that is, A is diagonalizable) and (2) $\mathfrak{g}^A$ is a Cartan subalgebra. It follows from Theorem 6.8 that a principal element $P \in G$ is regular.

Now a special relation, to be defined below, exists between the pair of

orthogonal Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ (See Theorem 6.7). We will show that this relation defines $\mathfrak{h}$ and $\mathfrak{h}'$ uniquely up to conjugacy.

Let $\mathfrak{h}_1$ and $\mathfrak{h}'_1$ be two Cartan subalgebras of $\mathfrak{g}$. We will then say that $\mathfrak{h}'_1$ is in appostition to $\mathfrak{h}_1$ with respect to a principal element $P \in G$ if (1) $\mathfrak{g}^P = \mathfrak{h}_1$ and, (2) $\mathfrak{h}'_1$ is stable under P and the set of eigenvalues of $P \mid \mathfrak{h}'_1$ includes a primitive s root of unity. (Observe that the number s has special significance for a principal element; the order of a principal element in G equals s.)

By Theorem 6.7 it is clear that $\mathfrak{h}'$ is in apposition to $\mathfrak{h}$ with respect to P_0.

THEOREM 7.3. *Assume that the Cartan subalgebra $\mathfrak{h}'_1$ is in apposition to $\mathfrak{h}_1$ with respect to the principal element P_1 and that $\mathfrak{h}'_2$ is in apposition to $\mathfrak{h}_2$ with respect to P_2. Then there exists $A \in G$ such that $AP_1A^{-1} = P_2$, $A\mathfrak{h}_1 = \mathfrak{h}_2$, and $A\mathfrak{h}'_1 = \mathfrak{h}'_2$.*

Proof. It suffices to assume $P_1 = P_0$, $\mathfrak{h}_1 = \mathfrak{h}$ and $\mathfrak{h}'_1 = \mathfrak{h}'$. Now all principal elements in G are conjugate to each other. Hence there exists $A_1 \in G$ such that $A_1P_0A_1^{-1} = P_2$. Since $\mathfrak{g}^{P_2} = \mathfrak{h}_2$, it is obvious that $A_1\mathfrak{h} = \mathfrak{h}_2$. Thus it also suffices to assume $\mathfrak{h}_2 = \mathfrak{h}$ and $P_2 = P_0$. But now $\mathfrak{h}'_2$ is stable under P_0 and $P_0 \mid \mathfrak{h}'_2$ has a primitive s root of unity as an eigenvalue. By Lemma 7.1 ω is also an eigenvalue of $P_0 \mid \mathfrak{h}'_2$. Let $y \in \mathfrak{h}'_2$, $y \neq 0$, be a corresponding eigenvector. Then in the notation of § 6.7, $y \in \mathfrak{u}_1$. But since y is contained in a Cartan subalgebra, y is not nilpotent. Hence by Lemma 7.3 there exists $A \in G$ and a non-zero scalar λ such that $y = \lambda Az_0$. But then by Corollary 6.4 y is regular and hence $A\mathfrak{h}' = \mathfrak{h}'_2$. Q. E. D.

8. The Coxeter-Killing transformation.

1. An element $\gamma' \in W$ will be called a transformation of Coxeter-Killing if it can be put in the form

$$\gamma' = R_{\alpha'_1}R_{\alpha'_2} \cdots R_{\alpha'_l},$$

where α'_i, $i = 1, 2, \cdots, l$, are the simple positive roots, in some order, relative to some Weyl chamber in $\mathfrak{h}^\#$. In recent years this transformation has been studied by Coxeter. Relationships between the eigenvalues of this transformation and the eigenvalues of the matrix of Cartan integers have been established. (See [5]). Also one knows that all the transformations of Coxeter-Killing in W form a conjugate class in W.

Let $\gamma = R_{\alpha_1}R_{\alpha_2} \cdots R_{\alpha_l}$. An integral valued function on the Weyl group,

which plays an important role in the cohomolgy theory of complex homogeneous spaces and also of π is the function $N(\sigma)$ which assigns to each $\sigma \in W$ the number of positive roots which go over into negative roots under the action of σ. The knowledge of $N(\gamma)$ will play an essential role in proving that the order of γ is s.

THEOREM 8.1. *Let γ be the transformation of Coxeter-Killing defined by letting $\gamma = R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_l}$. Then $N(\gamma) = l$. Furthermore if ϕ_i, $i = 1, 2, \cdots, l$, are the positive roots which change sign under γ then the roots ϕ_i form a basis of $\mathfrak{h}$.*

Proof. Let ϕ_i, $i = 1, 2, \cdots, l$, be defined by

$$\phi_i = R_{\alpha_1} R_{\alpha_{i-1}} \cdots R_{\alpha_{i+1}} \alpha_i.$$

It follows easily from (7.1.1) that

$$\phi_i = \alpha_i + \sum_{j>i} c_{ij} \alpha_j.$$

Since the coefficient of α_i in ϕ_i is one, it follows that ϕ_i is positive. Furthermore, since the coefficient of α_j in ϕ_i for $j < i$ is zero, it is clear that ϕ_i are linearly independent and hence form a basis of $\mathfrak{h}$.

But clearly

$$\gamma(\phi_i) = R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_{i-1}}(-\alpha_i)$$

and hence

$$\gamma(\phi_i) = -\alpha_i + \sum_{j<i} c'_{ij} \alpha_j.$$

Since the coefficient of α_i in $\gamma(\phi_i)$ is minus one, ϕ_i is a positive root which changes sign under γ. Thus $N(\gamma) \geqq l$.

Now it is well known that $N(R_{\alpha_i}) = 1$. In fact, α_i is the only positive root which changes sign under R_{α_i}. This is clear since only the coefficient of α_i in ϕ, for any $\phi \in \Delta$, is affected by R_{α_i}.

Now assume ϕ is a positive root which changes sign under γ. Since $\gamma(\phi) \in \Delta^-$, there clearly exists a maximal value i such that

$$R_{\alpha_i} R_{\alpha_{i+1}} \cdots R_{\alpha_l} \phi \in \Delta^-.$$

That is, $R_{\alpha_{i+1}} \cdots R_{\alpha_l} \phi \in \Delta^+$. It follows then that

$$\alpha_i = R_{\alpha_{i+1}} \cdots R_{\alpha_l} \phi.$$

This, however, means that $\phi = \phi_i$. Thus $N(\gamma) = l$. Q. E. D.

Now it is well known that one is not an eigenvalue of a Coxeter-Killing transformation. A proof of this resting on the work of Coxeter is given, for

example, in [4]. (See [4], p. 352). However, since a direct proof of this fact, making use of Theorem 8.1, can be given we shall include it.

LEMMA 8.1. *Let γ be the Coxeter-Killing transformation given as in* § 8.1. *Then one is not an eigenvalue of γ.*

Proof. Let ϵ_i, $i=1,2,\cdots,l$, be the basis of $\mathfrak{h}$ given as in § 5.1. Let $\epsilon'_i = 2\epsilon_i/(\alpha_i,\alpha_i)$. Clearly

$$\begin{aligned}\gamma\epsilon'_i &= R_{\alpha_1}\cdots R_{\alpha_i}\epsilon'_i \\ &= \epsilon'_i + R_{\alpha_1}\cdots R_{\alpha_{i+1}}(-\alpha_i)\end{aligned}$$

by (7.1.1). On the other hand, it is clear from the definition of ϕ_i in the proof of Theorem 8.1 that $R_{\alpha_1}\cdots R_{\alpha_{i+1}}(-\alpha_i) = \gamma(\phi_i)$. Thus we have

$$\gamma\epsilon'_i = \epsilon'_i + \gamma(\phi_i).$$

Assume now that $x \in \mathfrak{h}$ is fixed under γ. Write $x = \sum_{i=1}^{l} a_i\epsilon'_i$. Then

$$\begin{aligned}x &= \gamma x \\ &= \sum_{i=1}^{l} a_i(\epsilon'_i + \gamma(\phi_i)) \\ &= x + \sum_{i=1}^{l} a_i\gamma(\phi_i).\end{aligned}$$

Thus $\sum_{i=1}^{l} a_i\gamma(\phi_i) = 0$. However, by Theorem 8.1 the vectors ϕ_i and hence $\gamma(\phi_i)$ are linearly independent. Thus $a_i = 0$, $i=1,2,\cdots,l$, and hence $x=0$. Q. E. D.

8.2. Lemma 8.1 is needed solely to prove

LEMMA 8.2. *Let $\Gamma_i \subseteq \Delta$, $i=1,2,\cdots,L$, be the orbits in Δ under the action of γ. Then for $i=1,2,\cdots,L$*

$$\sum_{\phi\in\Gamma_i} \phi = 0.$$

Proof. This follows immediately from Lemma 8.1 since the sum of the roots in any orbit of γ is obviously left fixed by γ. Q. E. D.

The following theorem is proved in [4]

THEOREM 8.2 (Coleman). *Let h be the order of the Coxeter-Killing transformation γ defined as in* § 8.1. *Let $\nu = e^{2\pi i/h}$. Then there exists a regular eigenvector z_1 of γ whose corresponding eigenvalue is ν.*

Coleman also observed and used in [4] the following consequence of Theorem 8.2. We repeat his proof.

COROLLARY 8.2. *As in Lemma* 8.2 *let* $\Gamma_i \subseteq \Delta$, $i = 1, 2, \cdots, L$, *denote the distinct orbits of* γ *with respect to its action on the set of roots* Δ. *Then each orbit* Γ_i *contains exactly* h *roots so that in particular* $hL = 2r$, *where* $2r$ *is the total number of roots.*

Proof. Let $\phi \in \Delta$. Assume $\gamma^m \phi = \phi$. It suffices to prove that h divides m. Now, where z_1 is given as in Theorem 8.2,

$$\begin{aligned}(z_1, \phi) &= (z_1, \gamma^m \phi) \\ &= (\gamma^{-m} z_1, \phi) \\ &= \nu^{-m}(z_1, \phi).\end{aligned}$$

But since z_1 is regular, $(z_1, \phi) \neq 0$ and hence $\nu^{-m} = 1$. This implies h divides m. Q. E. D.

8.3. Now one knows that the Poincaré polynomial $P_G(t)$ can be put in the form

$$P_G(t) = (1 + t^{2m_1+1})(1 + t^{2m_2+1}) \cdots (1 + t^{2m_l+1}),$$

where the m_i, $i = 1, 2, \cdots, l$, are positive integers in non-decreasing order. The integers m_i (sometimes $m_i + 1$) are called the exponents of $\mathfrak{g}$ (or W as in [4]). When the values of the exponents for the simple exceptional Lie algebras were announced by Chevalley at the International Congress at Cambridge in 1950, Coxeter recognized a rather remarkable coincidence. He observed (1) that in all cases $hl = 2r$, so that in our notation $h = s$, and hence $\nu = \omega$ (See Corollary 6.8), (2) $m_i \leqq h$ and (3) the eigenvalues of γ are ω^{m_i}, $i = 1, 2, \cdots, l$. No proofs were given and the question of obtaining a proof remained open until recently. In [4], using $hl = 2r$ as the only empirically observed fact, Coleman proved that $m_i \leqq h$ and that the eigenvalues of a Coxeter-Killing transformation are indeed ω^{m_i}, $i = 1, 2, \cdots, l$. To make the proof independent of any empirical information Coleman states that it would be desirable to prove that $hl = 2r$, that is, to prove that the number L of orbits in Δ under the action of γ equals l. (See Corollary 8.2.) Another open question, stated in [4], is proving that $h = q + 1$, where q is the order of the highest root. (See [4], p. 356.) But since by definition $s = 1 + q$, it is clear from Corollary 6.8 that both of these questions are settled when it is shown that $h = s$. The statement that $h = s$ is a part of Theorem 8.4 below.

8.4. Now let γ be defined as in § 8.1. Let $A_\gamma \in G$ be any extension of γ. Let $\Gamma_i \subseteq \Delta$, $i = 1, 2, \cdots, L$, be defined as in § 8.2 and let

$$\mathfrak{b}_i = \sum_{\phi \in \Gamma_i} (e_\phi). \tag{8.4.1}$$

By Corollary 8.2 it is clear that each subspace $\mathfrak{b}_i$ is of dimension h. Furthermore, since

$$A_\gamma((e_\phi)) = (e_{\gamma\phi})$$

for any $\phi \in \Delta$, it is clear that

$$\mathfrak{g} = \mathfrak{h} + \sum_{i=1}^{L} \mathfrak{b}_i$$

is a direct sum decomposition of $\mathfrak{g}$ into subspaces which are stable under A_γ.

Now since γ^h is the identity element of W, we can define the scalar λ_ϕ, for any root ϕ, by the relation

$$(A_\gamma)^h e_\phi = \lambda_\phi e_\phi. \tag{8.4.2}$$

By applying A_γ to both sides of (8.4.2) it is clear that $\lambda_\phi = \lambda_{\gamma\phi}$ for all ϕ. Now since $A_\gamma{}^h$ reduces to the identity on $\mathfrak{h}$, we can write $A_\gamma{}^h = \operatorname{Exp} x$ for some $x \in \mathfrak{h}$. It follows that $\lambda_\phi = e^{(x,\phi)}$ for any $\phi \in \Delta$. But then

$$\begin{aligned}(\lambda_\phi)^h &= \prod_{i=0}^{h-1} \lambda_{\gamma^i\phi} \\ &= e^{(x,\phi+\gamma\phi+\cdots+\gamma^{h-1}\phi)} \\ &= 1\end{aligned}$$

by Lemma 8.2. Thus λ_ϕ is an h root of unity. This shows that $A_\gamma{}^{2h} = 1$. We can now prove that $h = s$ and hence that $L = l$. More than this we have

THEOREM 8.4. *Let γ be a Coxeter-Killing transformation on the Cartan subalgebra $\mathfrak{h}$. Let h be the order of γ. Then $h = 1 + \sum_{i=1}^{l} q_i$, where the integers q_i are the coefficients of the highest root relative to a basis of simple positive roots. That is, $h = s$. Moreover, $hl = 2r$, where r is the number of positive roots so that there are l distinct orbits $\Gamma_i \subseteq \Delta$, $i = 1, 2, \cdots, l$, in Δ under the action of γ and each orbit contains h roots. Furthermore, if we take $\gamma = R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_l}$ and let ϕ_i, $i = 1, 2, \cdots, l$, be the positive roots which change sign under γ (see Theorem 8.1), then we can choose an ordering of the orbits so that $\phi_i \in \Gamma_i$, $i = 1, 2, \cdots, l$.*

Now let $A_\gamma \in G$ be any extension of γ. Then $A_\gamma{}^h = 1$. That is, $A_\gamma{}^h e_\phi = e_\phi$ for any $\phi \in \Delta$ so that we can renormalize the root vectors e_ϕ in such a way that

$$A_\gamma{}^i e_\phi = e_{\gamma^i\phi}$$

for any $\phi \in \Delta$ and any integer i.

Now let $w_i \in \mathfrak{g}$, $i = 1, 2, \cdots, l$, be defined by

$$w_i = \sum_{\phi \in \Gamma_i} e_\phi$$

and let $\mathfrak{h}$ *be the subspace spanned by the elements* w_i. *Then* $\mathfrak{h}$ *is a Cartan subalgebra of* $\mathfrak{g}$ (*so that the elements* w_i *are semisimple and commute with each other*). *Furthermore,* $\bar{\mathfrak{h}} = \mathfrak{g}^{A_\gamma}$.

Proof. Let γ, A_γ and λ_ϕ for $\phi \in \Delta$ be as previously defined. It has been shown that ${A_\gamma}^{2h} = 1$. In particular A_γ is semisimple. But if $A \in G$ is semisimple it is well known that $\mathfrak{g}^A$ contains a Cartan subalgebra of $\mathfrak{g}$ (See e.g. [8]). For $A = A_\gamma$ the proof is somewhat more direct since A_γ, having finite order, lies in a maximal compact subgroup of G. But then $A_\gamma = \operatorname{Exp} x$ for some x in a compact form of $\mathfrak{g}$. We could then apply § 6.1(d)). Let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}^{A_\gamma}$.

Now let $\mathfrak{b}_i$, $i = 1, 2, \cdots, L$, be defined by (8.4.1). Consider the decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{i=1}^{L} \mathfrak{b}_i .$$

Obviously

$$\begin{aligned} \mathfrak{g}^{A_\gamma} &= \mathfrak{h} \cap \mathfrak{g}^{A_\gamma} + \sum_{i=1}^{L} \mathfrak{b}_i \cap \mathfrak{g}^{A_\gamma} \\ &= \sum_{i=1}^{L} \mathfrak{b}_i \cap \mathfrak{g}^{A_\gamma} \end{aligned} \tag{8.4.3}$$

by Lemma 8.1.

Now let $\phi \in \Gamma_i$. As we have already noted $\lambda_\phi = \lambda_{\gamma^i \phi}$ for any integer i. Thus ${A_\gamma}^h$ reduces to the scalar λ_ϕ on the space $\mathfrak{b}_i$. But then if $\lambda_\phi \neq 1$, ${A_\gamma}^h$ has no non-zero fixed vectors in $\mathfrak{b}_i$ and hence certainly $\mathfrak{b}_i \cap \mathfrak{g}^{A_\gamma} = 0$. On the other hand, if $\lambda_\phi = 1$, then clearly $\mathfrak{b}_i \cap \mathfrak{g}^{A_\gamma}$ is the one dimensional subspace spanned by $\sum_{i=0}^{h-1} {A_\gamma}^i e_\phi$. Thus if L_1 is the number of integers i, $1 \leqq i \leqq L$, such that $\lambda_\phi = 1$ for all $\phi \in \Gamma_i$, it follows from (8.4.3) that $L_1 = \dim \mathfrak{g}^{A_\gamma}$. In fact, since $\mathfrak{h} \subseteq \mathfrak{g}^{A_\gamma}$, we then have

$$L \geqq L_1 = \dim \mathfrak{g}^{A_\gamma} \geqq l$$

and hence in particular $L \geqq l$.

Now we assert that for any i, $\Gamma_i \cap \Delta^+$ and $\Gamma_i \cap \Delta^-$ are both non-empty. Indeed assume, without loss, that $\Gamma_i \cap \Delta^-$ is empty. Then $\sum_{\phi \in \Gamma_i} \phi$, in the lexicographical order of $\mathfrak{h}^\#$, must be strictly positive and hence cannot vanish. This contradicts Lemma 8.1 and hence the assertion is proved. Now since $\Gamma_i \cap \Delta^+$ and $\Gamma_i \cap \Delta^-$ are non-empty, it is clear that there exists a root $\phi \in \Gamma_i \cap \Delta^+$ such that $\gamma\phi \in \Gamma_i \cap \Delta^-$. That is, each orbit Γ_i contains at least one positive root which changes sign under γ. But then if we apply Theorem 8.1 it follows obviously that $L \leqq l$. Thus $L = L_1 = \dim \mathfrak{g}^{A_\gamma} = l$. That is, $\lambda_\phi = 1$ for all $\phi \in \Delta$ and $\mathfrak{g}^{A_\gamma} = \bar{\mathfrak{h}}$. Furthermore, if the root vectors e_ϕ, $\phi \in \Delta$,

are normalized so that $A_\gamma e_\phi = e_{\gamma\phi}$, it follows that the vectors $w_i = \sum_{\phi \in \Gamma_i} e_\phi$ form a basis of $\mathfrak{g}^{A_\gamma}$. Finally, observe that since $L = l$, each orbit Γ_i contains exactly one positive root which changes sign under γ. Q. E. D.

8.5. Let A_γ be as in Theorem 8.4. As we have noted in § 8.3 the eigenvalues of the restriction γ of A_γ to $\mathfrak{h}$ are the numbers ω^{m_i}, $i = 1, 2, \cdots, l$, where the m_i are the exponents of $\mathfrak{g}$. Let $\mathfrak{m}$ be the B-orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. It is clear that $\mathfrak{m}$ is stable under A_γ. Theorem 8.4 enables us to determine the eigenvalues of A_γ on $\mathfrak{m}$.

COROLLARY 8.5. *Let A be any extension of the Coxeter-Killing transformation γ and let the root vectors be normalized so that $Ae_\phi = e_{\gamma\phi}$ for all $\phi \in \Delta$. Let $\omega = e^{2\pi i/s}$ and let the roots ϕ_i be defined as in Theorem* 8.1. *Define for $i = 1, 2, \cdots, l$, $j = 1, 2, \cdots, s$,*

$$y^j{}_i = \sum_{k=0}^{h-1} \bar{\omega}^{jk} e_{\gamma^k \phi_i}.$$

Then $y^j{}_i$ is a basis of the B-orthogonal complement $\mathfrak{m}$ to the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and

$$A_\gamma y^j{}_i = \omega^j y^j{}_i$$

for all given values of i and j. In particular, the eigenvalues of A_γ on $\mathfrak{m}$ are ω^j, $j = 0, 1, \cdots, s-1$, and each eigenvalue occurs with multiplicity l.

Proof. This is an immediate consequence of Theorem 8.4 which asserts that A_γ on any of the subspaces $\mathfrak{b}_i$ permutes the basal elements $e_{\gamma^k \phi_i}$, $k = 1, 2, \cdots, s-1$, according to the cyclic permutation $(1, 2, \cdots, s)$. Q. E. D.

8.6. Upon comparing Corollary 8.5 with Theorem 6.8 the suggestion arises that perhaps A_γ is a principal element of $\mathfrak{g}$ and that $\mathfrak{h}$ is in apposition to $\mathfrak{g}^{A_\gamma}$ with respect to A_γ. This is in fact the case as Theorem 8.6 states.

As in Theorem 8.4 write $\tilde{\mathfrak{h}}$ for the Cartan subalgeba $\mathfrak{g}^{A_\gamma}$. Then since all the elements of $\tilde{\mathfrak{h}}$ are fixed by A_γ, there exists an element $w \in \tilde{\mathfrak{h}}$ such that

$$A_\gamma = \operatorname{Exp} w.$$

However, such an element w is not unique. We will have to choose w correctly in order to prove Theorem 8.6.

Let $A \in G$ be such that $A\mathfrak{h} = \mathfrak{h}$. Hereafter, the addition of the symbol $(\sim)$ to previous notation designates the effect of applying the automorphism A. Now let Y designate the infinite discrete group of translations in $\tilde{\mathfrak{h}}$ by all vectors of the form $\sum_{i=1}^{l} t_i \tilde{\epsilon}_i$, where the coefficients t_i are integers and ϵ_i

$i=1,2,\cdots,l$, are defined as in § 5.1. It is then easy to prove and well known fact that for x and $y\in\tilde{\mathfrak{h}}$, $\operatorname{Exp}2\pi ix=\operatorname{Exp}2\pi iy$ if and only if there exists $\delta\in Y$ such that $\delta x=y$.

For any $\sigma\in W$ and $\delta\in Y$ it is obvious that $\sigma\delta\sigma^{-1}\in Y$. It follows that if Z is the set of all linear transformations of $\tilde{\mathfrak{h}}$ of the form $\delta\sigma$, $\delta\in Y$, $\sigma\in\tilde{W}$, then Z is a group (a semi-direct product of Y and W). It is well known and not difficult to show that for any $x,y\in\tilde{\mathfrak{h}}$, $\operatorname{Exp}2\pi ix$ is conjugate by an element in $N(\tilde{H})$, the normalizer of $\tilde{H}$, to $\operatorname{Exp}2\pi iy$ if and only if there exists $\eta\in Z$ such that $\eta x=y$. (See [14], Theorem 6, p. 177).

Consider the action of Z on $\tilde{\mathfrak{h}}^{\#}$. Let

$$T=\{y\in\tilde{\mathfrak{h}}\mid(\tilde{\alpha}_i,y)\geqq 0,(\tilde{\psi},y)\leqq 1\text{ for }i=1,2,\cdots,l\}. \tag{8.6.1}$$

The set T is called the funadmental simplex of the chamber $\tilde{D}$. As one knows T has the property that given any $y\in\tilde{\mathfrak{h}}^{\#}$ there exists $x\in T$ and $n\in Z$ such that $\eta x=y$. (See [14], Theorem p. 177 and Theorem 8, p. 180). As a consequence, if we define

$$U=\bigcup_{\sigma\in W}\sigma T,$$

then since $Z=Y\tilde{W}$, given any $y\in\tilde{\mathfrak{h}}^{\#}$, there exists $x\in U$, $\delta\in Y$ such that $\delta x=y$. That is, there exists $x\in U$ such that $\operatorname{Exp}2\pi ix=\operatorname{Exp}2\pi iy$. Now we can write $A_\gamma=\operatorname{Exp}2\pi iy$ for some $y\in\tilde{\mathfrak{h}}$. However, since the eigenvalues of A_γ have modulus one it follows that (y,ϕ) must be real for every $\phi\in\tilde{\Delta}$. Thus y must be contained in $\tilde{\mathfrak{h}}^{\#}$. But then by what we have just seen we can choose $x\in U$ so that $A_\gamma=\operatorname{Exp}2\pi ix$. But then $x\in\sigma T$ for some $\sigma\in\tilde{W}$. Now our choice of A was arbitrary except only that $A\mathfrak{h}=\tilde{\mathfrak{h}}$. Without loss of generality then (modification of A) we can assume that $\sigma=1$. That is, for a suitable choice of A we have

$$A_\gamma=\operatorname{Exp}2\pi ix$$

for some $x\in T$.

Now by Theorem 8.4 all eigenvalues of A_γ are s roots unity. Thus if we write

$$x=\sum_{i=1}^{l}(b_i/s)\tilde{\epsilon}_i$$

then the scalars b_i are non-negative integers. But now since $(\tilde{\psi},x)\leqq 1$, it follows from (6.3.1) that

$$\sum_{i=1}^{l}q_ib_i/s\leqq 1. \tag{8.6.2}$$

But we have more information than this. By Theorem 8.4 A_γ is regular. Thus (ϕ,x) cannot be an integer for any $\phi\in\tilde{\Delta}$. Thus the strict inequality holds in (8.6.2). Furthermore, $b_i>0$. Hence we conclude

$$b_i\geqq 1$$

and

$$s > \sum_{i=1}^{l} q_i b_i.$$

But $s - 1 = q = \sum_{i=1}^{l} q_i$ (see (6.3.1)). Hence it follows that $b_i = 1$ for all i. That is,

$$x = \sum \tilde{\epsilon}_i / s$$

or

$$x = \tilde{x}_0 / s$$

But $\tilde{x}_0$ is a principal regular element of $\mathfrak{g}$ and

$$A_\gamma = \operatorname{Exp}(2\pi i/s)\tilde{x}_0.$$

Thus A_γ is a principal element of G. In fact $A_\gamma = APA^{-1}$. Furthermore, since the restriction γ of A_γ to $\mathfrak{h}$ has ω for an eigenvalue, it follows by definition (See § 7.3) that $\mathfrak{h}$ is in apposition to $\tilde{\mathfrak{h}}$ with respect to A_γ. We have proved

THEOREM 8.6. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let W be the Weyl group operating on $\mathfrak{h}$ and let $\gamma \in W$ be a Coxeter-Killing transformation. That is, $\gamma = R_{\alpha_1} R_{\alpha_2} \cdots, R_{\alpha_l}$, where α_i, $i = 1, 2, \cdots, l$, are simple positive roots relative to some lexicographical order in $\mathfrak{h}^\#$ and $R_{\alpha_i} \in W$ are the reflections they define.*

Let A_γ be any element of the adjoint group G of $\mathfrak{g}$ which extends γ. Then A_γ is a principal element of G (so that in particular its order is s and all such extensions are conjugate to each other). Let $\tilde{\mathfrak{h}}$ be the set of fixed elements of A_γ. Then $\tilde{\mathfrak{h}}$ is a Cartan subalgebra (since principal elements are regular) and $\mathfrak{h}$ is in apposition to $\tilde{\mathfrak{h}}$ with respect to A_γ (see § 7.3).

We derive a number of corollaries. The first is an immediate consequence of Theorem 7.3 and Theorem 8.6.

COROLLARY 8.6. *Let $\mathfrak{h}_1$ and $\mathfrak{h}'_1$ be Cartan subalgebras. Assume that $\mathfrak{h}'_1$ is in apposition to $\mathfrak{h}_1$ with respect to the principal element $P \in G$. Then the restriction of P to $\mathfrak{h}'_1$ defines a Coxeter-Killing transformation of $\mathfrak{h}'_1$. In particular, the restriction of P_0 to $\mathfrak{h}'$ is a Coxeter-Killing transformation (See Theorem 6.7).*

8.7. In § 9.2 we obtain a more general result (Theorem 9.2) than Corollary 8.6.

Taking § 2.5(h) into account the next corollary asserts the validity of the empirical method found by A. Shapiro for the determination of the exponents m_i.

COROLLARY 8.7. *For $i = 1, 2, \cdots, l$ let k_i be as given as in § 6.5 and*

let m_i be the exponents of $\mathfrak{g}$ in non-decreasing order. Then $k_i = m_i$, $i = 1, 2, \cdots, l$.

In other words, if d_i, $i = 1, 2, \cdots, l$, are the dimensions of the irreducible components of the adjoint representation of a principal TDS *on $\mathfrak{g}$, then*

$$\prod_{i=1}^{l} (1 + t^{d_i})$$

is the Poincaré polynomial of G.

Proof. This is an immediate consequence of the theorem of Coleman (see § 8.3), Theorem 8.4, Theorem 6.7 and Corollary 8.6. Q.E.D.

The proof of Theorem 8.6 yields a characterization of principal elements of G. We know that a principal element of G is regular and that its order is s. Among all regular elements in G we now show that its order is minimal.

COROLLARY 8.6. *Let $A \in G$ be regular and let k be its order (possibly ∞). Then $k \geqq s$, where $s = 1 + q$ and q is the order of the highest root. Furthermore, $k = s$ if and only if A is a principal element of G.*

Proof. It suffices to assume the order k of A is finite. Now $\mathfrak{g}^A$ is a Cartan subalgebra. Without loss we may assume this to be $\mathfrak{h}$. Since A has finite order the eigenvalues of A have modulus 1. Thus we may write $A = \operatorname{Exp} 2\pi i x$, where $x \in \mathfrak{h}^{\#}$. More than this, as argued in the proof of Theorem 8.6, by conjugating A, if necessary, we may asume x is contained in the fundamental simplex of the chamber D. That is, $(x, \alpha_i) \geqq 0$, $i = 1, 2, \cdots, l$, $(x, \psi) \leqq 1$. On the other hand, since A is regular, (x, ϕ) is not an integer for any $\phi \in \Delta$ so that strict inequalities hold in the inequalities just given. Thus we may write $x = \sum_{i=1}^{l} (t_i/k)\epsilon_i$, where the t_i are positive integers and $\sum_{i=1}^{l} t_i q_i < k$. But then

$$s - 1 = \sum_{i=1}^{l} q_i \leqq \sum_{i=1}^{l} t_i q_i < k$$

which proves $s \leqq k$. Also if $k = s$ then $t_i = 1$ for $i = 1, 2, \cdots, l$. But in such a case A is clearly principal. In fact $A = P_0$.

9. A theorem on automorphisms and a characterization of cyclic elements.

1. Let $S^*(\mathfrak{g})$ be the algebra of all polynomials on $\mathfrak{g}$ (the symmetric algebra over the dual space to $\mathfrak{g}$). We may write $S^*(\mathfrak{g}) = \sum_{k=0}^{\infty} S^k(\mathfrak{g})$, where

$S^k(\mathfrak{g})$ is a space of homogeneous polynomials of degree k. The group G operates on $S^*(\mathfrak{g})$ by the relation

$$A(F)(x) = F(A^{-1}x), \tag{9.1.1}$$

where $A \in G$, $F \in S^*(\mathfrak{g})$, $x \in \mathfrak{g}$. Let $J^*(\mathfrak{g}) \subseteq S^*(\mathfrak{g})$ be the algebra of polynomials which are left fixed by all $A \in G$ and let $J^k(\mathfrak{g}) = S^k(\mathfrak{g}) \cap J^*(\mathfrak{g})$. Then it is due to Chevalley that $J^*(\mathfrak{g})$ is generated by l algebraically independent homogeneous polynomials I_j, $j = 1, 2, \cdots, l$. Furthermore, if $I_j \in J^{p_j}(\mathfrak{g})$, where $p_1 \leqq p_2 \leqq \cdots \leqq p_l$, then $p_i = m_i + 1$. (See [3] and also [2].)

The following lemma is an immediate consequence of this result of Chevalley and Corollary 8.7.

LEMMA 9.1. *Let I_j, $j = 1, 2, \cdots, l$, be the invariant polynomials defined as above. Then $\deg I_l = s$ and $\deg I_j < s$ for all $j < l$.*

Now let $J^*_+(\mathfrak{g})$ be the hyperplane in $J^*(\mathfrak{g})$ consisting of all polynomials in $J^*(\mathfrak{g})$ with zero constant term. Consider the variety in $\mathfrak{g}$ formed by the zeros of all polynomials in $J^*_+(\mathfrak{g})$. As one might expect,

THEOREM 9.1. *Let $x \in \mathfrak{g}$. Then $F(x) = 0$ for all $F \in J^*_+(\mathfrak{g})$ if and only if x is nilpotent.*

Proof. Let $F_k \in S^k(\mathfrak{g})$ be defined by

$$F_k(y) = \operatorname{trace}(\operatorname{ad} y)^k.$$

It is clear that $F_k \in J^k(\mathfrak{g})$. It is also clear that if x is not nilpotent $F_k(x) \neq 0$ for some integer k. Hence to prove Theorem 9.1 it must be shown, conversely, that if x is nilpotent, $F(x) = 0$ for all $F \in J^*(\mathfrak{g})$. Assume then that x is nilpotent. By Lemma 3.3 there exists $y \in \mathfrak{g}$ such that $[y, x] = x$. Let λ be an arbitrary non-zero real scalar and let $A^\lambda = \operatorname{Exp} \lambda y$. Clearly $A^\lambda x = e^\lambda x$. Now let $F \in J^k(\mathfrak{g})$. Obviously by (9.1.1)

$$F(x) = F(A^\lambda x) = F(e^\lambda x) = e^{k\lambda} F(x).$$

But $e^{k\lambda} \neq 1$. Thus $F(x) = 0$. This proves Theorem 9.1 since $J^*(\mathfrak{g})$ is generated by homogeneous polynomials. Q. E. D.

9.2. In the proof of Theorem 9.1 we have seen that if x is nilpotent and c is any non-zero scalar, there exists an automorphism $A \in G$ such that $Ax = cx$. The situation as Theorem 9.2 indicates is entirely different when x is not nilpotent. If x is cyclic, it has been shown (see Theorem 6.7) that there exists a principle element $P \in G$ such that $Px = \mu x$, where μ is a primitive

s root of unity. Among all possible relations of the form $Ax = cx$, where $A \in G$, and x is not nilpotent it will be shown that this is, in a very definite way, the critical case.

THEOREM 9.2. *Assume $A \in G$, $x \in \mathfrak{g}$ and $Ax = cx$. Assume further that x is not nilpotent. Then c is a primitive m-th root of unity for some integer m. Furthermore, m divides $m_j + 1$ for some exponent m_j of $\mathfrak{g}$ so that in particular $m \leqq s$. Finally, if $m = s$, then* (1) *x is a cyclic element of $\mathfrak{g}$ and* (2) *A is a principal element of G. Moreover, if $\mathfrak{h}_1$ is the Cartan subalgebra which contains x (cyclic implies regular), then $\mathfrak{h}_1$ is stable under A and the restriction σ of A to $\mathfrak{h}_1$ is a Coxeter-Killing transformation.*

Proof. Since x is not nilpotent, it follows from Theorem 9.1 that there exists j, $1 \leqq j \leqq l$, such that $I_j(x) \neq 0$. But

$$I_j(x) = I_j(Ax) = I_j(cx) = c^{p_j} I_j(x).$$

Thus $c^{p_j} = 1$. This proves that c is a primitive m-th root of unity, where m is an integer which divides $p_j = m_j + 1$. By Lemma 9.1, $m < s$.

Now let $S^*(\mathfrak{h})$ be the algebra of polynomials on $\mathfrak{h}$ and let $J^*(\mathfrak{h})$ be the subalgebra of all polynomials which are fixed under the action of the Weyl group W. Let $\tilde{\chi}\colon S^*(\mathfrak{g}) \to S^*(\mathfrak{h})$ be the homomorphism defined by restricting a polynomial on $\mathfrak{g}$ to $\mathfrak{h}$; then it is a known result of Chevalley that the restriction

$$\chi\colon J^*(\mathfrak{g}) \to J^*(\mathfrak{h}) \tag{9.2.1}$$

of $\tilde{\chi}$ to $J^*(\mathfrak{g})$ defines an isomorphism of $J^*(\mathfrak{g})$ onto $J^*(\mathfrak{h})$. (See [3], § IV; also [12], Theorème 2, p. 19-10).

To prove the remaining part of Theorem 9.2 we shall need

LEMMA 9.2. *Let $I_j \in J^*(\mathfrak{g})$, $j = 1, 2, \cdots, l$, be given as in § 9.1. Let $x, y \in \mathfrak{h}$. Then x and y are conjugate with respect to W if and only if $I_j(x) = I_j(y)$ for $j = 1, 2, \cdots, l$.*

Proof. It is obvious from the definition of W that if x and y are conjugate with respect to W, $I_j(x) = I_j(y)$ for $j = 1, 2, \cdots, l$. Conversely, assume $I_j(x) = I_j(y)$, $j = 1, 2, \cdots, l$. Let $x_1, x_2, \cdots, x_{r_1}$ and $y_1, y_2, \cdots, y_{r_2}$ be, respectively, the distinct conjugates of x and y under W. Assume x is not conjugate to y. Then x_i, y_j, $i = 1, 2, \cdots, r_1$, $j = 1, 2, \cdots, r_2$, represent $r_1 + r_2$ points in $\mathfrak{h}$. Since points in $\mathfrak{h}$ are obviously algebraic varieties there exists $F \in S^*(\mathfrak{h})$ such that $F(x_i) = 0$, $i = 1, 2, \cdots, r_1$, $F(y_j) = 0$, $j = 2, \cdots, r_2$, and $F(y_1) = 1$. But then if

$$\tilde{F} = \sum_{\sigma \in W} \sigma(F),$$

it follows that $\tilde{F} \in J^*(\mathfrak{h})$, $\tilde{F}(y) \neq 0$ whereas $F(x) = 0$. But $\chi(I_j)$, $j = 1, 2, \cdots, l$, generate $J^*(\mathfrak{h})$ (see (9.2.1). Thus there exists j such that $I_j(x) \neq I_j(y)$. This is a contradiction. Q. E. D.

Proof of Theorem 9.2 *continued.* Assume $m = s$. Then $c^{p_j} = 1$ implies $j = l$ by Lemma 9.1. Thus we conclude $I_j(x) = 0$ for $j = 1, 2, \cdots, l-1$ and $I_l(x) \neq 0$.

Now it is well known (e.g. see [9], Theorem 6) that we may write $x = x_1 + x_2$, uniquely, where x_1 is semi-simple, x_2 is nilpotent and $[x_1, x_2] = 0$. By the uniqueness of this decomposition it is clear that $Ax_1 = cx_1$ and $Ax_2 = cx_2$. Furthermore, by hypothesis we know that $x_1 \neq 0$. Thus if we apply the previous argument to the relation $Ax_1 = cx_1$ it follows that $I_j(x_1) = 0$ for $j = 1, 2, \cdots, l-1$, and $I_l(x_1) \neq 0$.

Now since x_1 is semi-simple it lies in a Cartan subalgebra. Without loss of generality we may assume $x_1 \in \mathfrak{h}$. Now let γ be the Coxeter-Killing transformation given as in § 8.1. By Corollary 8.6, Theorem 7.3 and Theorem 6.7 there exists a cyclic vector $z \in \mathfrak{h}$ such that $\gamma z = \omega z$. If we apply the argument above once more to the relation $A_\gamma z = \omega z$, where A_γ is any extension of γ, it follows that $I_j(z) = 0$ for $j = 1, 2, \cdots, l-1$ and $I_l(z) \neq 0$. But then by Lemma 9.2, x_1 must be conjugate to a scalar multiple of z. Thus x_1 is cyclic. But then x_1 is regular. Since $[x_1, x_2] = 0$ this implies $x_2 \in \mathfrak{h}$. But x_2 is nilpotent. Hence $x_2 = 0$ or $x = x_1$. Thus x is cyclic.

Now since the regular vector $x \in \mathfrak{h}$ is an eigenvector of A it follows that $\mathfrak{h}$ is stable under A. Let $\tau \in W$ be the restriction of A to $\mathfrak{h}$. By Lemma 7.1 ω is an eigenvalue of τ. Let $y \in \mathfrak{h}$ be the corresponding eigenvector. The argument above applied to y proves that y is cyclic and when properly normalized there exists $\sigma \in W$ such that $\sigma z = y$. Now let $\gamma_1 = \sigma\gamma\sigma^{-1}$. Then γ_1 is a Coxeter-Killing transformation and $\gamma_1 y = \omega y$. But then $\tau^{-1}\gamma_1$ leaves y fixed. But it is well known that the only element of W which leaves a regular element fixed is the identity (see [8]). Thus τ is a Coxeter-Killing transformation and since A is an extension of τ, it follows from Theorem 8.6 that A is a principal element of G. Q. E. D.

Observe that Theorem 9.2 is a generalization of Corollary 8.6.

In [4] an important role is played by a regular eigenvectaor of γ. It is easy to see (see [4]) that the eigenvalue corresponding to any regular eigenvector of γ is a primitive s root of unity. By Theorem 9.2 we see that all such eigenvectors are cyclic.

Corollary 9.2 characterizes a Coxeter-Killing transformation among all elements in W.

COROLLARY 9.2. *Let $\sigma \in W$ be arbitrary and let c be an eigenvalue of σ.*

Then c is a primitive m-th root of unity, where m is an integer which divides m_j+1 for some exponent m_j of $\mathfrak{g}$. In particular $m \leqq s$. Moreover, if $m=s$, then σ is a Coxeter-Killing transformation and the eigenvector of σ corresponding to c is a cyclic element of $\mathfrak{g}$.

9.3. A corollary of the proof of Theorem 9.2 yields a characterization of cyclic elements in $\mathfrak{g}$.

COROLLARY 9.2. *Let I_j, $j=1,2,\cdots,l$, be the invariant polynomials on $\mathfrak{g}$ given as in § 9.1. Let $x \in \mathfrak{g}$. Then x is cyclic if and only if*

$$I_j(x)=0 \qquad \text{for } j=1,2,\cdots,l-1$$

and

$$I_l(x) \neq 0.$$

Proof. The proof that the equations above hold when x is cyclic has been given in the proof of Theorem 9.2. Assume, conversely, that these equations hold. Write

$$x=x_1+x_2, \tag{9.3.1}$$

where x_1 is semi-simple, x_2 is nilpotent and $[x_1,x_2]=0$. (See [9], Theorem 6). By Theorem 9.1 $x_1 \neq 0$. Without loss of generality we may assume $x_1 \in \mathfrak{h}$. Consider $\mathfrak{g}^{x_1}$. It is clear that $\mathfrak{g}^{x_1}$ is generated linearly by $\mathfrak{h}$ and all root vectors e_ϕ, where $(\phi, x_1)=0$. It follows easily that $\mathfrak{g}^{x_1}=\mathfrak{c}+[\mathfrak{g}^{x_1},\mathfrak{g}^{x_1}]$ is a direct sum (see [9], Theorem 7), where $\mathfrak{c}$ is the center of $\mathfrak{g}^{x_1}$. Furthermore, $[\mathfrak{g}^{x_1},\mathfrak{g}^{x_1}]$ is semi-simple and $\mathfrak{c} \subseteq \mathfrak{h}$. Now $x_2 \in \mathfrak{g}^{x_1}$. Write $x_2=y_1+y_2$, where $y_1 \in \mathfrak{c}$ and $y_2 \in [\mathfrak{g}^{x_1},\mathfrak{g}^{x_1}]$. It follows immediately that y_2 is a nilpotent element of $[\mathfrak{g}^{x_1},\mathfrak{g}^{x_1}]$ and hence y_2 is a nilpotent element of $\mathfrak{g}$. (See footnote 3.)

But $x_1+y_1 \in \mathfrak{h}$ is semi-simple. Writing $x=(x_1+y_1)+y_2$ gives a second decomposition of the form (9.3.1). By uniqueness $y_1=0$ and hence $x_2 \in [\mathfrak{g}^{x_1},\mathfrak{g}^{x_1}]$. Applying Lemma 3.3 there exists $y \in [\mathfrak{g}^{x_1},\mathfrak{g}^{x_1}]$ such that $[y,x_2]=x_2$. Of course $[y,x_1]=0$. Thus if $A^\lambda=\operatorname{Exp}\lambda y$, it is clear that

$$A^\lambda x=x_1+e^\lambda x_2.$$

But for any $F \in J^*(\mathfrak{g})$, $F(x)=F(A^\lambda x)$. By choosing λ such that e^λ is arbitrarily close to 0 it follows from continuity that $F(x)=F(x_1)$. In particular, $I_j(x_1)=0$ for $j=1,2,\cdots,l-1$ and $I_l(x_1) \neq 0$. But then as in the proof of Theorem 9.2 it follows that x_1 is cyclic. But then x_1 is regular and hence $[x_1,x_2]=0$ implies $x_2 \in \mathfrak{h}$. Since x_2 is nilpotent this implies $x_2=0$ and hence $x=x_1$ is cyclic. Q. E. D.

Even without knowledge of Corollary 9.2 it is interesting to observe that the conditions of that corollary, which singles out I_l, is independent of the set of generators I_j of $J^*(\mathfrak{g})$ so long as they are homogeneous, alge-

braically independent and ordered the way they are. This would not be necessarily true if I_j was singled out where $j < l$.

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[14] H. Weyl, *The structure and representation of continuous groups*, Institute for Advanced Study notes by R. Brauer, 1934-1935, reprinted 1955.

(*Added in proof:* The empirical procedure for determining the exponents of $\mathfrak{g}$, discovered by A. Shapiro, was recently found independently by R. Steinberg who likewise speaks about the lack of any proof for the procedure. See § 9 in

[15] R. Steinberg, "Finite reflection groups," *Transactions of the American Mathematical Society*, vol. 91 (1959), pp. 493-504.

As we have noted in the Introduction a proof (see Corollary 8.7) of the procedure is a consequence of our main results. The paper [15] is mainly devoted to giving a proof of two empirical discoveries of H. S. M. Coxeter. A proof of these discoveries is also given here (see Theorem 8.4)).

ON THE CONNECTEDNESS THEOREM IN ALGEBRAIC GEOMETRY.*[1]

By WEI-LIANG CHOW.

To Zariski on his 60th birthday.

1. Introduction. The Connectedness Theorem in abstract algebraic geometry was first formulated and proved by Zariski [14]; it stems from his attempt to prove the Principle of Degeneration over an abstract field and actually contains that principle as a special case. The proof of Zariski makes extensive use of his theory of holomorphic functions on an algebraic variety; in fact, this latter theory was developed by him specially for this purpose, and, as has been pointed out by Zariski himself, it has in a sense the Connectedness Theorem as its principal application. In view of the fundamental importance of the Connectedness Theorem on the one hand and the very complicated nature of Zariski's theory of holomorphic functions on the other, it is clearly very desirable to have for this theorem a proof of a simpler and more elemetnary nature. Furthermore, in consonance with the recent tendencies in algebraic geometry toward increasing arithmetic applications, one would naturally wish to generalize this theorem to the case of a correspondence between the rational transforms of arbitrary local domains (the exact meaning of this statement will be explained later in § 2). In a recent paper [4] we have given a very simple proof of such a generalization for a special case of the theorem, namely the Principle of Degeneration. In the present paper we shall solve the problem completely by giving a simple proof of the Connectedness Theorem in just about as general a version as we know how to formulate it at this stage of the subject.

Let R be a Noetherian domain with the quotient field K, and let S^n be the projective space of dimension n in the algebraic geometry over a universal domain $\mathfrak{R}$ which contains the field K. Let $\mathfrak{p}$ be a prime ideal in R, and denote by $\bar{R}$ the residue field $R/\mathfrak{p}$ of R over $\mathfrak{p}$; generally for any element x in R or a polynomial $f(Y)$ in $R[Y] = R[Y_0, Y_1, \cdots, Y_m]$, where the Y_i are indeterminates, we shall denote by $\bar{x}$ or $\bar{f}(X)$ respectively the corresponding element

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in $\bar{R}$ or polynomial in $\bar{R}[Y]$ under the canonical homomorphism of R onto $\bar{R}$. We shall denote by $\bar{S}^n$ the projective space of dimension n in the algebraic geometry over a universal domain Λ which contains the field $\bar{R}$. Let Z be a positive r-cycle in S^n such that its associated point $y = (y_0, y_1, \cdots, y_m)$ in S^m is rational over K. A point η in $\bar{S}^m$ is said to be a *specialization* of the point y at the prime ideal $\mathfrak{p}$ if whenever a form $f(Y)$ satisfies the condition $f(y) = 0$, it satisfies also the condition $\bar{f}(\eta) = 0$. Since the condition for a point to be an associated point is "universal," every specialization η of y at $\mathfrak{p}$ is the associated point of a positive r-cycle $Z(\eta)$ in $\bar{S}^n$, of the same degree as Z; we shall say that the cycle $Z(\eta)$ is a *specialization* of the cycle Z at $\mathfrak{p}$ or at $R_{\mathfrak{p}}$, where $R_{\mathfrak{p}}$ is the quotient ring of R with respect to $\mathfrak{p}$. It is easily seen that the set of all specializations of y at $\mathfrak{p}$ is an $\bar{R}$-closed subset in $\bar{S}^m$, and it follows from this that the union of the supports of all specializations of Z at $\mathfrak{p}$ is also an $\bar{R}$-closed subset in $\bar{S}^n$. Recalling the definition that R is *analytically irreducible* at $\mathfrak{p}$ if the completion of the local ring $R_{\mathfrak{p}}$ is an integral domain, we can state the Connectedness Theorem as follows:

Let R be a Noetherian domain with the quotient field K, and let Z be a rational positive cycle over K in S^n; if Z is connected and if $\mathfrak{p}$ is a prime ideal in R such that R is analytically irreducible at $\mathfrak{p}$, then union of the supports of all the specializations of Z at $\mathfrak{p}$ is also connected.

Apart from the extension to an arbitrary Noetherian domain (instead of the coordinate ring of an algebraic variety), the above version of the Connectedness Theorem coincides in substance with the original formulation of Zariski. Now, since the Connectedness Theorem concerns itself essentially with the properties of a cycle Z with respect to a prime ideal $\mathfrak{p}$ in R, there will be no loss of generality if we replace the ring R by the quotient ring $R_{\mathfrak{p}}$, so that we can assume from the beginning that R is already a local domain. Moreover, since the concept of analytic irreducibility involves the completion of R, we may just as well assume further that R is itself a complete local domain. It turns out that, once put in this strictly local form, the Connectedness Theorem can be generalized to a theorem in the "relative" geometry, in a similar way as we have generalized the Principle of Degeneration in [4]. This generalization, which we shall call the General Connectedness Theorem, can be stated as follows:

GENERAL CONNECTEDNESS THEOREM. *Let R be a complete local domain with the quotient field K and the residue field $\bar{R}$, and let Z be a rational positive cycle over K in S^n; if Z is K-connected, then the union of the supports of all the specializations of Z at R is $\bar{R}$-connected.*

We remark that here the topological concepts such as "closed," "connected," etc., refer to the well-known Zariski topology, in which the closed subsets are unions of finite number of algebraic varieties, and the relative topological concepts such as "K-closed," "K-connected," etc., refer to the relative Zariski topology, in which the closed subsets are unions of finite number of complete sets of conjugate varieties over K. The use of these topological terminology, though not really necessary, proves to be very convenient. Although it is really only a trivial observation, we should like to point out in this connection that our formulation of the General Connectedness Theorem does not at all imply that the preservation of connectedness under specialization, which holds in the absolute Zariski topology, holds also in the relative Zariski topology over a *given* ground field; on the contrary, a little reflection shows that it is rather the disconnectedness over a ground field that is preserved under a specialization over the same ground field.

The General Connectedness Theorem contains two special cases: The Special Connectedness Theorem, where the cycle Z consists of a single point, so that we are dealing essentially with what corresponds to a birational transformation in the geometric case, and the Principle of Degeneration, where the local domain R is a real discrete valuation ring, so that there is a uniquely determined specialization of Z at R. Conversely, once we have proved these two special cases, we can easily obtain from them the Connectedness Theorem, as we shall show later in section 4. However, in order to prove the General Connectedness Theorem in its full generality as formulated above, we need something more than a simple combination of these two special cases. The reason for this is not difficult to see. For the hypothesis of the General Connectedness Theorem provides only the relative connectedness of the cycle Z over the complete local domain R, while the application of the Principle of Degeneration (in the form formulated in [4]) presupposes that we have some information about the relative connectedness of Z over some complete real discrete valuation ring dominating R, and it is clear that the relative connectedness of a cycle is in general not preserved under such an extension. In order to fill this gap, we need a result which says essentially that if a cycle Z splits into two disconnected parts over every complete real discrete valuation ring dominating R and this is in "continuous" manner, then Z cannot be connected over R. It turns out that these considerations lead us in a natural way to a refinement of the General Connectedness Theorem, which we shall call (for reasons which will be apparent) the Extended Principle of Degeneration. We shall prove this Extended Principle of Degeneration only for the case of a complete regular local ring; whether this Extend Principle

holds for the general case of an arbitrary complete local domain, is a question which will not be settled here, although it seems very likely that the answer to it is in the affirmative. However, so far as the General Connectedness Theorem is concerned, this restriction to regular local rings is not a serious point, for a very simple argument (in section 4) shows that the validity of this theorem for a regular local ring implies its validity in the general case. On the other hand, this restriction to regular local rings enables us to obtain, in case of the Special Connectedness Theorem, yet another refinement, which is not generally true without this restriction. We shall say that a point set is *linearly connected* if every two points in the set can be connected by a sequence of rational curves in the set. Then we can prove that in the Special Connectedness Theorem for a regular local ring the set of all specializations is not only connected, but also linearly connected. This stronger result, which we shall call the Linear Connectedness Theorem, is of fundamental importance for the theory of rational equivalence, as we shall show in our forthcoming work on the intersection theory of cycles and equivalence classes of cycles.

Our paper therefore consists of two main results, the Linear Connectedness Theorem and the Extended Principle of Degeneration; the former will be proved in section 3 and the latter will be proved in section 5. Our proof of the Linear Connectedness Theorem is based on an analysis of the properties of the quadratic transforms of a regular local ring, and we shall use in course of the proof some of the well-known results on this subject. Looked upon from this point of view, our proof is related to the recent proofs of Zariski [15] and Murre [10] of the Special Connectedness Theorem for a simple point in the geometric case. On the other hand, our proof of the Extended Principle of Degeneration is of a quite different nature; although the quadratic transforms play also an essential role in this proof, the main idea lies in a different direction and is essentially a generalization of the method used in our proof of the Principle of Degeneration in [4], which centers on the idea of Hensel's Lemma. Only, as we have to deal now with an entire system of specialization cycles at the center, instead of a single cycle, the elementary "resultant" method used in [4] proves to be too cumbersome and rather inadequate; instead, we shall draw more heavily on the theory of local rings, with the result that we are able not only to prove the more general result, but also to gain a better insight into the nature of the problem. Incidentally, we should like to point out that the result in [4] is not used in our present proof, which in this sense can be said to start from the beginning, while on the other hand our recent result on the Local Bertini Theorem [5] plays an essential role in the important process of reduction to the case of dimension 2.

As to terminology, we shall use here some of the terms in Weil's well-known *Foundations of Algebraic Geometry*, such as the terms "universal domain," "variables," "indeterminates," "cycles," etc., which have now become more or less standard; however, we shall not make use of any results in this book, so that a knowledge of it is not at all necessary for the understanding of the present paper. On the other hand, we shall assume that the reader is familiar with the theory of local rings, which is fundamental for our subject; we shall therefore use freely the basic notions and results in this theory, although we endeavor to give explicit references for all results beyond the very elementary ones. We shall also use for convenience the topological concepts based on the Zariski topology, either absolute or relative to a given ground field; the Zariski topology for algebraic varieties is of course involved right at the beginning already in the formulation of our problem, but later we shall introduce it also for sets of algebraic cycles as well as for sets of valuation rings dominating given local domains. Finally, in the last two sections we shall assume some knowledge of the theory of associated forms; it is sufficient for this purpose to know the properties of the associated forms listed at the beginning of § 4 in our paper [4].

2. Rational transforms. We begin by making a few remarks concerning the projective space S^n (or S^m) introduced in the preceding section. Although it is quite possible for us to work within a given universal domain provided we choose it suitably to include all the local rings we shall need in the course of our theory, it is nevertheless more convenient and probably in a sense also more appropriate to consider S^n as the "universal projective space" which can be defined as the logical union of all the projective spaces of dimension n over all possible universal domains. We remark however that we shall never invoke more than a finite number of universal domains at one time, so that we are not really involved in any of the logical difficulties usually connected with the use of the expression "all" in such a context. Without any reference to any universal domains, we can define the *universal projective space* S^n by the stipulations that any ordered system of $n+1$ elements $(z_0, z_1, \cdots, z_n)$ in any field (or integral domain), not all zero, defines a point z in S^n, and that two such systems define the same point in S^n if and only if their elements are contained in some one field and are proportional to each other in this field; the elements z_j are called the (homogeneous) coordinates of the point z. We shall say that the point z is rational over a field K if the ratios of its coordinates z_j are contained in K; when this is the case, we shall always assume (unless stated to the contrary) that the pro-

portionality factor has already been so chosen that the coordinates of z are all in K. In applying the usual algebro-geometrical operations to the entities in this universal projective space S^n we must of course see to it that the entities involved all belong to some one field, while among the entities belonging to different fields only very restricted types of relations can occur.

On the other hand, the projective space $\bar{S}^n$ remains, as it was before, to be the one over a given universal domain Λ, so that we can now consider $\bar{S}^n$ to be contained in the universal projective space S^n. However, this space $\bar{S}^n$ is in a sense the center of our attention, and we shall relate to it various entities in S^n in the following way. We shall say that a local ring $(R, \mathfrak{p})$ is *attached* to Λ if its residue field $\bar{R} = R/\mathfrak{p}$ is identified with a field contained in Λ, whereby the expression "a field in Λ" is to be understood in the sense of Weil's *Foundations* (i.e. Λ has infinite degree of transcendency over $\bar{R}$); the so defined concept of an attached local ring is somewhat more specific than that of a local ring by itself, for two local rings which are isomorphic but attached to distinct fields in Λ are to be considered here as different. *In this paper all the local rings are assumed to be attached to* Λ *in some ways*, so that we shall not always explicitly mention this fact. Furthermore, if a local ring $(R_1, \mathfrak{p}_1)$ contains R such that $R \cap \mathfrak{p}_1 = \mathfrak{p}$, and if R_1 is attached to Λ in such a way that the canonical embedding of $R/\mathfrak{p}$ into $R_1/\mathfrak{p}_1$ as abstract fields is accompanied by a corresponding embedding of the attached residue field of R in Λ into the attached residue field of R_1 in Λ, then we shall say that R_1 *dominates* R. In particular, we shall always assume that the completion R^* of R is attached to Λ in such a way that R^* dominates R, so that they have the same residue field in Λ. Once a local ring R is attached to Λ, the canonical homomorphism of R onto $\bar{R}$ will induce (under certain circumstances) a relation between those geometrical entities in S^n which are defined by means of R and similar entities in $\bar{S}^n$, such as the concept of a specialization of a cycle at $\mathfrak{p}$ described in the preceding section. For the sake of completeness and clarity, we shall begin here anew and repeat all the definitions.

Let $(R, \mathfrak{p})$ be a local domain of dimension t (attached to Λ), with the quotient field K and the residue field $\bar{R}$; for any element a in R or a polynomial $f(Y)$ in $R[Y] = R[Y_0, Y_1, \cdots, Y_m]$, we shall denote by $\bar{a}$ or $\bar{f}(X)$ respectively the corresponding element in $\bar{R}$ or polynomial in $\bar{R}[Y]$ under the canonical homomorphism of R onto $\bar{R}$. Let $y = (y_0, y_1, \cdots, y_m)$ be a point in S^m with coordinates in a field containing K; a point $\eta = (\eta_0, \eta_1, \cdots, \eta_m)$ in $\bar{S}^m$ is said to be a *specialization of* y *at* R, if for every form $f(Y)$ in $R[Y]$ the relation $f(y) = 0$ implies the relation $\bar{f}(\eta) = 0$. We shall denote by $\bar{y}(R)$ the set of all specializations of y at R, and it is easily seen that $\bar{y}(R)$

is an $\bar{R}$-closed subset in $\bar{S}^m$. If $(R_1, \mathfrak{p}_1)$ is a local domain which dominates $(R, \mathfrak{p})$ and if y is a point which is rational over a field containing R_1, then every specialization of y at R_1 is also a specialization of y at R, so that $\bar{y}(R_1)$ is contained in $\bar{y}(R)$. In case y is the associated point of a positive r-cycle Z in S^n, then every specialization of y at R is also the associated point of a positive r-cycle in $\bar{S}^n$, which we shall call a *specialization of Z at R*; we shall denote by $\bar{Z}(R)$ the set of all specializations of Z at R and denote by $|\bar{Z}(R)|$ the union of the supports of all the cycles in $\bar{Z}(R)$, and we observe that $|\bar{Z}(R)|$ is an $\bar{R}$-closed subset in $\bar{S}^n$.

We shall be particularly interested in the case where the point y is rational over K, and we shall say in this case that the point y defines a *rational transform* $y(R)$ of the local domain R; as we shall see, this notion of a rational transform of a local domain plays a fundamental role in our theory. The set $\bar{y}(R)$ is then called the *center* of the rational transform $y(R)$. If η is any point in $\bar{y}(R)$, consider the subring $\mathfrak{Q}(y(R)/\eta)$ in K consisting of all the elements of the form $f(y)/g(y)$, where $f(Y)$ and $g(Y)$ are forms of the same degree in $R[Y]$ and $\bar{g}(\eta) \neq 0$; it is easily seen that $\mathfrak{Q}(y(R)/\eta)$ is a local domain with the quotient field K, and that its maximal prime ideal $\mathfrak{p}(y(R)/\eta)$ consists of all the elements $f(y)/g(y)$ with $\bar{f}(\eta) = 0$. Since the residue field of $\mathfrak{Q}(y(R)/\eta)$ is isomorphic to the field $\bar{R}(\eta)$ in Λ, we shall attach $\mathfrak{Q}(y(R)/\eta)$ to Λ by identifying the residue field of $\mathfrak{Q}(y(R)/\eta)$ with $\bar{R}(\eta)$, and we shall call the so attached local domain $\mathfrak{Q}(y(R)/\eta)$ the *specialization ring* of the rational transform $y(R)$ at η. We observe that $\mathfrak{Q}(y(R)/\eta)$ evidently dominates the local domain R, and that if $\eta_j \neq 0$ for any one j, then $\mathfrak{Q}(y(R)/\eta)$ contains also the ring $R[y_0/y_j, \cdots, y_m/y_j]$; furthermore, it is easily seen that $\bar{y}(Q(y(R)/\eta))$ is the $\bar{R}$-variety in $\bar{S}^{m-1}$ consisting of all the specializations of η over $\bar{R}$. The point η is said to be *normal* or *simple* in the rational transform $y(R)$ if $\mathfrak{Q}(y(R)/\eta)$ is integrally closed or regular respectively, and when such is the case, we shall also say that the rational transform $y(R)$ is normal or simple at η respectively. We shall say that the rational transform $y(R)$ is normal or non-singular if it is normal or simple respectively at every point in its center $\bar{y}(R)$.

The concept of a rational transform $y(R)$ of a local domain R is a natural generalization of the concept of a rational transformation T of an algebraic variety V around a given point p of the variety. It belongs to the local geometry in the strict sense that one is concerned with what corresponds to the behavior of the rational transformation T in the infinitesimal neighborhood of p in V, not a finite neighborhood, however small. For this reason the only points on our rational transform $y(R)$ are the points in the center $\bar{y}(R)$, which

corresponds in the geometric case to the total image $T(p)$ of the point p under T. However, we should like to stress here that we are really considering the points in $\bar{y}(R)$ not as points in what corresponds to the image variety $T(V)$ of V, but rather as points in what corresponds to the graph of T in the product space $V \times T(V)$, so that $\bar{y}(R)$ should be considered as corresponding to $p \times T(p)$, not to $T(p)$; for our definition of the specialization ring $\mathfrak{Q}(y(R)/\eta)$ at a point η in $\bar{y}(R)$ corresponds to the specialization ring of the graph of T at a point in $p \times T(p)$, not that of $T(V)$ at a point of $T(p)$. The reason for introducing here what corresponds to the graph rather than the image is quite easily understood, for the graph of a rational transformation contains essentially the complete information about the transformation, while the image is merely its projection on one of the factor spaces and hence its introduction is often not necessary, as is the case in our present studies.

The study of rational transforms is of fundamental importance not only for our present purpose, but also for many other problems in abstract local geometry. In this section, we shall limit ourselves to stating and proving a few basic properties of these transforms, which we shall need later. The discerning reader will notice that most of these are merely generalizations to arbitrary local domains of some of the elementary properties given in [11] for the geometric case, and it is therefore not surprising that we have also taken over some of the proofs from that paper.

LEMMA 2.1. *If $y(R)$ is a rational transform of R, then for any two points η and η' in $\bar{y}(R)$, $\mathfrak{Q}(y(R)/\eta)$ contains $\mathfrak{Q}(y(R)/\eta')$ if and only if η' is a specialization of η over $\bar{R}$.*

Proof. The condition is evidently sufficient. To prove the necessity, let $g(Y)$ be any form in $R[Y]$ such that $\bar{g}(\eta') \neq 0$, and let $f(Y)$ be a form in $R[Y]$ of the same degree as $g(Y)$ such that $\bar{f}(\eta) \neq 0$; if $\mathfrak{Q}(y(R)/\eta)$ contains $\mathfrak{Q}(y(R)/\eta')$, then $f(y)/g(y)$ being an element in $\mathfrak{Q}(y(R)/\eta')$ must be contained in $\mathfrak{Q}(y(R)/\eta)$. Since $\bar{f}(\eta) \neq 0$, this implies that $\bar{g}(\eta) \neq 0$. Since this is true for any form $g(Y)$ in $R[Y]$ and since any form in $\bar{R}[Y]$ can be obtained as a form $\bar{g}(Y)$, this shows that η' is a specialization of η over $\bar{R}$.

In the special case when the local domain R is a valuation ring, the center $\bar{y}(R)$ of any rational transform $y(R)$ of R consists of a single point; in fact, there exists in this case some one y_j such that $y_0/y_j, \cdots, y_m/y_j$ are all contained in R, and the point $(\overline{y_0/y_j}, \cdots, \overline{y_m/y_j})$ is evidently the only specialization of y at R. Furthermore, it is easily seen from the well-known maximality properties of a valuation ring that in this case we have $\mathfrak{Q}(y(R)/\bar{y}(R)) = R$.

In the general case of an arbitrary local domain R, we consider a valuation ring M in K dominating R; then, for any rational transform $y(R)$ of R, the point $\bar{y}(M)$ is evidently contained in $\bar{y}(R)$. We shall call this point $\bar{y}(M)$ the *center* of M in the rational transform $y(R)$. We maintain that M dominates $\mathfrak{L}(y(R)/\bar{y}(M))$. To see this, let y_j be so chosen that the elements $y_0/y_j, \cdots, y_m/y_j$ are all in M and hence $\bar{y}(M) = (\overline{y_0/y_j}, \cdots, \overline{y_m/y_j})$; if $f(Y)$ is any form of degree d in $R[Y]$, then $f(y)/y_j^d$ is an element in M, and since M dominates R, it is a non-unit in M if and only if $\bar{f}(\bar{y}(M)) = 0$. It follows easily from this that any element $f(y)/g(y) = (f(y)/y_j^d)/(g(y)/y_j^d)$ in $\mathfrak{L}(y(R)/\bar{y}(M))$ is an element in M, and that it is a non-unit in M if and only if it is such in $\mathfrak{L}(y(R)/\bar{y}(M))$. This result, together with Lemma 2.1, gives us almost immediately the following lemma.

LEMMA 2.2. *Let $y(R)$ be a rational transform of a local domain R, let M be a valuation ring in K which dominates R, and let η be a point in $\bar{y}(R)$; then $\bar{y}(M) = \eta$ if and only if M dominates $\mathfrak{L}(y(R)/\eta)$.*

It is well-known that any local ring in a field K is dominated by some valuation ring in K. Applying this to the local ring $\mathfrak{L}(y(R)/\eta)$, we conclude from Lemma 2.2 that given any point η in $\bar{y}(R)$ there exists a valuation ring M in K dominating R such that $\eta = \bar{y}(M)$. If we denote by $\mathfrak{V}(R)$ the *set of all valuation rings in K dominating R*, then the correspondence $M \to \bar{y}(M)$ defines a mapping $\mathfrak{V}(R)$ *onto* the center $\bar{y}(R)$ of $y(R)$. We now introduce a topology in $\mathfrak{V}(R)$, called its *Zariski topology*, by the stipulation that a subset in $\mathfrak{V}(R)$ is said to be closed if it is the inverse image under this mapping of a closed subset in $\bar{y}(R)$ for some one rational transform $y(R)$ of R, whereby $\bar{y}(R)$ is understood to be endowed with its Zariski topology (induced by the Zariski topology in $\bar{S}^m$). If we consider only the Zariski $\bar{R}$-topology of $\bar{y}(R)$ for every $y(R)$, then we obtain the $\bar{R}$-topology of $\mathfrak{V}(R)$, and we have therefore such notions as $\bar{R}$-closed, $\bar{R}$-connected, etc. That this definition of a closed subset in $\mathfrak{V}(R)$ actually defines a topology in $\mathfrak{V}(R)$ can be easily verified. We observe also that this topology is the least fine among all topologies in $\mathfrak{V}(R)$ for which the mapping $M \to \bar{y}(M)$ is continuous for every rational transform $y(R)$ of R, both in the absolute case and in the relative case.

In case y is the associated point of a positive cycle Z in S^n, then for every valuation ring M in K dominating R we have a specialization cycle $\bar{Z}(M)$ of Z at R, and in this way we can obtain all specializations of Z at R, so that we can set $\bar{Z}(R) = \{\bar{Z}(M) \mid M \in \mathfrak{V}(R)\}$. If $F(U)$ is the associated form of the cycle Z, then we shall denote the associated form of the

specialization cycle $\bar{Z}(M)$ by $\bar{F}_M(U)$. We note that both forms $F(U)$ and $\bar{F}_M(U)$ are defined only up to proportionality factors in K and $\bar{M}$ respectively, and that even when $F(U)$ is taken to be a form in $M[U]$, $\bar{F}_M(U)$ is not necessarily the residue image of $F(U)$ in $\bar{M}[U]$; in order to get $\bar{F}_M(U)$ as the residue form of $F(U)$, one has to normalize $F(U)$ by dividing it by one of its coefficients which have the lowest value under the valuation of M, so that all coefficients in $F(U)$ be in M but not all non-units.

Let $z=(z_0, z_1, \cdots, z_l)$ be a point in S^l which is rational over K, and consider the rational transform $z(R)$; then we can define a birational correspondence T between the two rational transforms $y(R)$ and $z(R)$ as follows: Two points η and ζ in $\bar{y}(R)$ and $\bar{z}(R)$ respectively are said to correspond to each other under T if there exists a valuation ring M in K dominating R such that $\eta=\bar{y}(M)$ and $\zeta=\bar{z}(M)$. In order to avoid possible ambiguities, we shall consider the correspondence T as from $y(R)$ to $z(R)$, so that the inverse correspondence from $z(R)$ to $y(R)$ will be denoted by T^{-1}; if η or ζ is any point in $\bar{y}(R)$ or $\bar{z}(R)$ respectively, then we shall denote by $T(\eta)$ or $T^{-1}(\zeta)$ respectively the set of all points in $\bar{z}(R)$ or $\bar{y}(R)$ which correspond to η or ζ under T or T^{-1} respectively. As the reader will readily perceive, this definition of a birational correspondence is essentially the same as that given in [11]. We shall say that the birational correspondence T is *regular* at a point η in $\bar{y}(R)$, if there exists a point ζ in $\bar{z}(R)$ corresponding to η under T such that $\mathfrak{Q}(y(R)/\eta)$ contains $\mathfrak{Q}(z(R)/\zeta)$. Finally, we introduce the *join* $y\circ z$ of the two points y and z, which is defined as the point in $S^{(m+1)(l+1)-1}$ whose coordinates are the elements $y_i z_j$, for $i=0, 1, \cdots, m$, and $j=0, 1, \cdots, l$, arranged in some arbitrary (but fixed) order; the rational transform $y\circ z(R)$ is then said to be the *join* of the rational transforms $y(R)$ and $z(R)$. It is clear that if M is any valuation ring in K dominating R, then we have the relation $\overline{y\circ z}(M)=\bar{y}(M)\circ\bar{z}(M)$; it follows easily from this that the birational correspondence between $y\circ z(R)$ and $y(R)$ is regular at every point in $\overline{y\circ z}(R)$.

LEMMA 2.3. *If the birational correspondence T between $y(R)$ and $z(R)$ is regular at a point η in $y(R)$, then $T(\eta)$ consists of exactly one point.*

Proof. By definition there exist a point ζ in $\bar{z}(R)$ and a valuation ring M in K dominating R such that $\eta=\bar{y}(M)$, $\zeta=\bar{z}(M)$, and $\mathfrak{Q}(y(R)/\eta)\supset\mathfrak{Q}(z(R)/\zeta)$. According to Lemma 2.2, M dominates both $\mathfrak{Q}(y(R)/\eta)$ and $\mathfrak{Q}(z(R)/\zeta)$; it follows that $\mathfrak{Q}(y(R)/\eta)$ dominates $\mathfrak{Q}(z(R)/\zeta)$. If M_1 is any valuation ring in K dominating $\mathfrak{Q}(y(R)/\eta)$, then it must also dominate $\mathfrak{Q}(z(R)/\zeta)$; it follows then by Lemma 2.2 again that $\bar{z}(M_1)=\zeta$, which proves our lemma.

LEMMA 2.4. *If $y(R)$ is normal at a point η and if $T(\eta)$ consists of a finite number of points, then T is regular at η.*

Proof. It is easily seen that, without affecting the validity of our lemma, we can replace $z(R)$ by another rational transform of R which is in biregular correspondence with it; in particular, we can replace $z(R)$ by the rational transform defined by a point whose coordinates are the monomials of the elements $z_0, z_1, \cdots, z_l$ of a given degree, arranged in some arbitrary order. By making such a replacement of $z(R)$ if necessary, we can therefore obtain that if $\zeta^{(1)}, \cdots, \zeta^{(d)}$ are the points contained in $T(\eta)$, then none of the coordinates $\zeta^{(1)}{}_0, \cdots, \zeta^{(d)}{}_0$ vanishes. Then $\mathfrak{Q}(z(R)/\zeta^{(i)})$ contains the ring $R[z_1/z_0, \cdots, z_l/z_0]$ for every $i = 1, \cdots, d$, and is in fact the quotient ring of $R[z_1/z_0, \cdots, z_l/z_0]$ with respect to the ideal

$$\mathfrak{q}_i = \mathfrak{p}(z(R)/\zeta^{(i)}) \cap R[z_1/z_0, \cdots, z_l/z_0].$$

Since each $\mathfrak{Q}(z(R)/\zeta^{(i)})$ is contained in some one valuation ring in $\mathfrak{V}(\mathfrak{Q}(y(R)/\eta))$, it follows that $R[z_1/z_0, \cdots, z_l/z_0]$ is contained in the intersection of all valuation rings in $\mathfrak{V}(\mathfrak{Q}(y(R)/\eta))$. Since $y(R)$ is normal at η, $\mathfrak{Q}(y(R)/\eta)$ coincides with the intersection of all valuation rings in $\mathfrak{V}(\mathfrak{Q}(y(R)/\eta))$ and hence contains $R[z_1/z_0, \cdots, z_l/z_0]$. Since $\mathfrak{q}_i$ is the restriction to $R[z_1/z_0, \cdots, z_l/z_0]$ of the maximal ideal in some one valuation ring in $\mathfrak{V}(\mathfrak{Q}(y(R)/\eta))$, it must contain the ideal

$$\mathfrak{q} = \mathfrak{p}(y(R)/\eta) \cap R[z_1/z_0, \cdots, z_l/z_0],$$

and since $\mathfrak{Q}(y(R)/\eta)$ contain the quotient ring of $R[z_1/z_0, \cdots, z_l/z_0]$ with respect to $\mathfrak{q}$, it must also contain $\mathfrak{Q}(z(R)/\zeta^{(i)})$. This shows that T is regular at the point η.

We make a digresion here to observe that up to now we have not used at all the crucial property of a local domain, that it is Noetherian; in fact, all our definitions (except for the definition of a simple point) and results up to now hold for any (commutative) integrity domain with the property that the set of all non-units is an ideal, i.e., what is sometimes called a quasi-local domain. While any deeper study of the rational transforms must of course involve the Noetherian or other similar properties, it is nevertheless convenient to avail ourselves of the fact that some of the elementary notions and results hold for the more general case. This is the justification for stating Lemma 2.2 above for any arbitrary valuation ring M, without the Noetherian restriction. In this paper we shall only consider rational transforms of (Noetherian) local domains; however; in considering the valuation rings dominating a local domain, we shall not restrict ourselves to the

Noetherian ones. Although it is possible in principle to get along without using any other valuation rings except those which are discrete and of rank 1, as shown by Lemma 2.7 below, it is very desirable to be able to use others whenever it turns out to be convenient to do so. We shall meet later occasions when it is convenient to use discrete valuation rings of rank 2.

From now on the Noetherian property of the local domain R will play an essential role. We observe first that by virtue of this Noetherian property the local domain R has a finite dimension t, which can be defined either as the rank of the maximal prime ideal $\mathfrak{p}$ or the number of elements in a system of parameters.

LEMMA 2.5. *If $y(R)$ is a rational transform of a local domain of dimension t, then any point in $\bar{y}(R)$ has a dimension at most $t-1$ over $\bar{R}$, so that the maximum dimension of the algebraic set $\bar{y}(R)$ is at most $t-1$.*

Proof. Let η be any point in $\bar{y}(R)$, and let M be a valuation ring in K dominating $\mathfrak{Q}(y(R)/\eta)$. According to [2], Theorem 1, the residue field of M has a dimension at most $t-1$ over $\bar{R}$; hence the residue field $\bar{R}(\eta)$ of $\mathfrak{Q}(y(R)/\eta)$ can have a dimension at most $t-1$ over $\bar{R}$, which shows that η has a dimension at most $t-1$ over $\bar{R}$.

LEMMA 2.6. *Let $y(R)$ be a rational transform of a local domain of dimension t; if a point η in $\bar{y}(R)$ has the dimension $t-1$ over $\bar{R}$, then there are only a finite number of valuation rings in K which dominate $\mathfrak{Q}(y(R)/\eta)$, and these valuation rings are real discrete (i.e. regular local domains of dimension 1).*

Proof. Since the residue field $\bar{R}(\eta)$ of $\mathfrak{Q}(y(R)/\eta)$ has the dimension $t-1$ over $\bar{R}$, the dimension of $\mathfrak{Q}(y(R)/\eta)$ must be 1; for, if the dimension of $\mathfrak{Q}(y(R)/\eta)$ were greater than 1, then there would exist a valuation ring in K dominating $\mathfrak{Q}(y(R)/\eta)$ whose residue field has a positive dimension over $\bar{R}(\eta)$ and hence a dimension greater than $t-1$ over $\bar{R}$, in contradiction to [2], Theorem 1. According to [7], the integral closure R_1 of $\mathfrak{Q}(y(R)/\eta)$ is Noetherian and hence the ideal $R_1\mathfrak{p}(y(R)/\eta)$ has only a finite number of (minimal) prime divisors; furthermore, the quotient ring of R_1 with respect to any one of these prime divisors is a real discrete valuation ring whose residue field is a finite algebraic extension of $\bar{R}(\eta)$. Since any valuation ring in K dominating $\mathfrak{Q}(y(R)/\eta)$ must coincide with one of these quotient rings, our lemma follows if we observe in addition that since the residue field of such a valuation ring is a finite algebraic extension of $\bar{R}(\eta)$, there are only a finite number of possible ways to embed it as a field extension of $\bar{R}(\eta)$ in Λ.

LEMMA 2.7. *Let $y(R)$ be a rational transform of an unmixed (= equidimensional) local domain of dimension t, and let η be any point in $y(R)$; then there exists a real discrete valuation ring in K which dominates $\mathfrak{Q}(y(R)/\eta)$ and whose residue field has the dimension $t-1$ over $\bar{R}$.*

Proof. Without the last condition on the dimension over R this lemma holds for any arbitrary local domain R (not necessarily unmixed); one has only to apply to $\mathfrak{Q}(y(R)/\eta)$ the general result that any local domain in K is dominated by a real discrete valuation ring in K, as proved in [1] or [4]. Furthermore, a closer look at the proof of this result in [4] shows that the residue field of the so obtained real discrete valuation ring has a dimension over the residue field $\bar{R}(\eta)$ of $\mathfrak{Q}(y(R)/\eta)$ which is exactly equal to the dimension of $\mathfrak{Q}(y(R)/\eta)$. The last condition in our lemma will then be fulfilled if we have the "dimension formula": $\dim(\mathfrak{Q}(y(R)/\eta)) + \dim(\eta/\bar{R}) = t$. This "dimension formula," in fact in a somewhat more general form, has been proved recently by Nagata [10] under the assumption that R satisfies the "second chain condition," and a previous result of Nagata shows that a unmixed local domain always satisfies this "second chain condition."

As a corollary to Lemma 2.7, we have the following useful lemma.

LEMMA 2.8. *If R is an analytically irreducible local domain, then $\bar{y}(R) = \bar{y}(R^*)$ for every rational transform $y(R)$ of R.*

Proof. Since R^* dominates R, it is clear that $\bar{y}(R^*) \subset \bar{y}(R)$; we have therefore only to show that every point in $\bar{y}(R)$ is also a point in $\bar{y}(R^*)$. Let η be a point in $\bar{y}(R)$ and let M be a real discrete valuation ring in K which dominates $Q(y(R)/\eta)$ and whose residue field has the dimension $t-1$ over $\bar{R}$ (t being the dimension of R). According to [13], Theorem 2, R is a subspace in M, so that the completion R^* of R can be embedded in the completion M^* of M; it is clear that M^* dominates both R^* and M. It follows that $\bar{y}(M^*)$ is a point in $\bar{y}(R^*)$ and also that $\bar{y}(M^*) = \bar{y}(M) = \eta$; this shows that η is a point in $y(R^*)$.

Before proceeding further, we shall make an observation here which will be useful to us also later. Let $v = (v_1, \cdots, v_d)$ be a set of independent variables over R, and consider the quotient ring $R(v)$ of $R[v]$ with respect to the prime ideal $R[v]\mathfrak{p}$; it is easily seen that $R(v)$ is a local domain of dimension t, with the quotient field $K(v)$, and that $R(v)$ is normal or regular if and only if R is such. Furthermore, if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in R, then we have $R(v)(\mathfrak{a} \cap \mathfrak{b}) = R(v)\mathfrak{a} \cap R(v)\mathfrak{b}$ and $R(v)\mathfrak{a} \cap R = \mathfrak{a}$, and if $\mathfrak{q}$ is a primary ideal in R with the associated prime ideal $\mathfrak{q}'$, then $R(v)$ is primary with

$R(v)\mathfrak{q}'$ as its associated prime. Finally, the $R(v)\mathfrak{p}$-residues $\bar{v} = (\bar{v}_1, \cdots, \bar{v}_d)$ of v are algebraically independent over $\bar{R}$, so that we can attach $R(v)$ to Λ by identifying the $\bar{v}$ with any set of d independent variables over $\bar{R}$; therefore, $R(v)$ dominates R and its residue field $\bar{R}(\bar{v})$ is a purely transcendental extension of dimension d over $\bar{R}$. For proof, we refer to [6], Lemma 14 for a part of it, with the remark that the rest of our assertions follows easily from it. In many problems these properties will enable us to replace the original local domain R by $R(v)$, and such a substitution often turns out to be convenient, mainly on account of the fact that the residue field $\bar{R}$ is thereby extended by the adjunction of an arbitrary number of variables and in particular made infinite. We shall call $R(v)$ the local domain obtained from R by the *adjunction of the variables* v.

Assume now that R is a complete regular local domain of dimension $t > 2$, and let $x_1, \cdots, x_t$ be the elements in a minimal base for $\mathfrak{p}$; then $R(v)$ is a regular local domain of dimension t and the elements $x_1, \cdots, x_t$ form also a minimal base for its maximal prime ideal $R(v)\mathfrak{p}$. Let $d \leqq t-2$, and set $u_i = (x_2 + v_i x_{i+2})/x_1$ for $i = 1, \cdots, d$; denote by R_v the quotient ring of $R(v)[u_1, \cdots, u_d]$ with respect to the prime ideal $R(v)[u_1, \cdots u_d]\mathfrak{p}$. It is easily seen that R_v is a regular local domain of dimension $t-d$, and that the $R_v\mathfrak{p}$-residues $\bar{u} = (\bar{u}_1, \cdots, \bar{u}_d)$ of $u = (u_1, \cdots, u_d)$ are algebraically independent over $\bar{R}(\bar{v})$, so that we can attach R_v to by identifying $\bar{u}$ with any set of d independent variables over $\bar{R}(\bar{v})$. The following lemma, which can be considered as the local equivalent of the well-known "Theorem of Bertini" in algebraic geometry, plays an essential role later in the proof of Theorem 2 in that it enables us to reduce that theorem to the case $t = 2$.

LEMMA 2.9. *The quotient field K of R is algebraically closed in the quotient field of the completion R_v^* of R_v.*

This lemma is equivalent to the assertion that if a monic polynomial of one variable with coefficients in R is irreducible over R, then it is also irreducible over the completion R_v^* of R_v. We can evidently without any loss of generality restrict ourselves to the case $d = 1$, the general case following then by induction, and for this case the assertion can be proved in exactly the same way as the lemma in [5], except that one replaces the element c there by the element $v = v_1$ and observes that v satisfies the condition of being "sufficiently general" as required there. We therefore refer the proof of this lemma to [5].

3. Quadratic transforms and linear connectedness theorem. Let

$(R,\mathfrak{p})$ be a local domain of dimension t, with the quotient field K and the residue field $\bar{R}$, and let $x_1,\cdots,x_s$ be elements which form a minimal basis for $\mathfrak{p}$; if we denote by x the point $(x_1,\cdots,x_s)$ in S^{s-1}, then the rational transform $x(R)$ of R is called the *quadratic transform* of R. Since any two minimal bases for $\mathfrak{p}$ have the same number of elements and can be obtained each from the other by a linear transformation with coefficients in R and with determinant not in $\mathfrak{p}$, the quadratic transform $x(R)$ of R is unique up to an everywhere biregular transformation. We shall be mainly interested in the case where the local domain R is regular, and we recall that in this case the number s is equal to the dimension t. In this case, it is well-known ([1], Lemma 3, and Corollary 1) that $\bar{x}(R)=\bar{S}^{t-1}$ and every point in $\bar{x}(R)$ is simple in $x(R)$; furthermore ([2], Lemma 10, or as a special case of the dimension formula in [10]), if ξ is any point in $\bar{x}(R)$ and if t' is the dimension of $\mathfrak{L}(x(R)/\xi)$, then $t'+\dim(\xi/\bar{R})=t$. In case ξ is a generic point of $\bar{x}(R)=\bar{S}^{t-1}$ over $\bar{R}$, then $\mathfrak{L}(x(R)/\xi)$ is a real discrete valuation ring, which (considered without its attachment to Λ) is the valuation ring of what is sometimes called the $\mathfrak{p}$-*adic valuation of* R; we shall call this valuation ring the *quadratic valuation ring* of R.

Let $y(R)$ be a rational transform of R in S^m, and let η be a point in $\bar{y}(R)$; let $\mathfrak{P}(\eta)$ be the homogeneous ideal in $R[Y]=R[Y_0,Y_1,\cdots,Y_m]$ generated by all forms $f(Y)$ such that $\bar{f}(\eta)=0$, and let h be a positive integer such that there exists a basis of this ideal $\mathfrak{P}(\eta)$ consisting of forms of degrees less than $h+1$. Let $b_1(Y),\cdots,b_l(Y)$ be a basis of the R-module of all forms of degree h in $\mathfrak{P}(\eta)$, and denote the point $(b_1(y),\cdots,b_l(y))$ by b; if we set $z=y\circ b$, then the rational transform $z(R)$ is called the *quadratic transform of* $y(R)$ *centered at* η. This definition is analogous to the one given in [11], II, § 11 (for a monoidal transformation), and by a similar argument as given there, we can show that the quadratic transform $z(R)$ is up to an everywhere biregular birational correspondence independent of the choice of the integer h and the basis $b_1(Y),\cdots,b_l(Y)$, being dependent only on the point η. Furthermore, it is clear that $z(R)$ remains unchanged if we replace the point η by any one of its generic specializations over $\bar{R}$; this shows that the concept of a quadratic transform belongs really to the relative geometry.

Let T be the birational correspondence between $y(R)$ and its quadratic transform $z(R)$ centered at η. It is clear from the definition $z=y\circ b$ that the inverse T^{-1} is everywhere regular in $\bar{z}(R)$, and that T itself is regular at every point in $\bar{y}(R)$ which is not a specialization of η over $\bar{R}$. As to the point η itself, we can take any valuation ring M in K dominating $\mathfrak{L}(y(R)/\eta)$

and get a point $\bar{z}(M) = \eta \circ \bar{b}(M)$ in $T(\eta)$, and in this way we can obtain every point in $T(\eta)$; it follows that $T(\eta) = \bar{z}(\mathfrak{L}(y(R)/\eta)) = \eta \circ \bar{b}(\mathfrak{L}(y(R)/\eta))$. Let $w_1, \cdots, w_\tau$ be the elements of a minimal basis for the maximal prime ideal in $\mathfrak{L}(y(R)/\eta)$, and denote by w the point $(w_1, \cdots, w_\tau)$ in $S^{\tau-1}$. It is easily seen that the coordinates of b, after division by the form y_i^h if $\eta_i \neq 0$, are linear combinations of the coordinates of w with coefficients in $\mathfrak{L}(y(R)/\eta)$, and vice-versa; therefore, the coordinates of z, after division by the form y_i^h, are also linear combinations of the coordinates of $y \circ w$ with coefficients in $\mathfrak{L}(y(R)/\eta)$, and vice-versa. Since both

$$\mathfrak{L}(z(R)/\bar{z}(M) \text{ and } \mathfrak{L}(y \circ w(R)/\eta \circ \bar{w}(M))$$

contain $\mathfrak{L}(y(R)/\eta)$ as a subring, it follows easily from this that

$$\mathfrak{L}(z(R)/\bar{z}(M)) = \mathfrak{L}(y \circ w(R)/\eta \circ \bar{w}(M));$$

furthermore, going over to the residue field of M, we conclude that the coordinates of $\bar{z}(M) = \eta \circ \bar{b}(M)$ and the coordinates of $\overline{y \circ w}(M) = \eta \circ \bar{w}(M)$ are linear combinations of each other with coefficients in $\bar{R}(\eta)$. This shows that the birational correspondence between $z(R)$ and $y \circ w(R)$ is biregular between $\bar{z}(\mathfrak{L}(y(R)/\eta))$ and $\overline{y \circ w}(\mathfrak{L}(y(R)/\eta))$, and that each $\bar{R}(\eta)$-variety in $T(\eta)$ is biregularly, in fact projectively equivalent over $\bar{R}(\eta)$ to an $\bar{R}(\eta)$-variety in $\eta \circ \bar{w}(\mathfrak{L}(y(R)/\eta))$.

LEMMA 3.1. *For any valuation ring M in K dominating $\mathfrak{L}(y(R)/\eta)$, we have the relation* $\mathfrak{L}(z(R)/\bar{z}(M)) = \mathfrak{L}(w(\mathfrak{L}(y(R)/\eta))/\bar{w}(M))$.

Proof. Every element in $\mathfrak{L}(w(\mathfrak{L}(y(R)/\eta))/\bar{w}(M))$ is the quotient $f(w)/g(w)$ of two forms of the same degree in w with coefficients in $\mathfrak{L}(y(R)/\eta)$, with the condition $\bar{g}(\bar{w}(M)) \neq 0$, while every element in $\mathfrak{L}(y(R)/\eta)$ is the quotient $f'(y)/g'(y)$ of two forms of the same degree in y with coefficients in R, with the condition $\bar{g}'(\eta) \neq 0$; it follows that, by multiplying both $f(w)$ and $g(w)$ with a suitable power of a coordinate y_i or w_j such that the i-th coordinate of η or the j-th coordinate of $\bar{w}(M)$ does not vanish respectively, we can express any element in $\mathfrak{L}(w(\mathfrak{L}(y(R)/\eta))\bar{w}(M))$ as a quotient $f''(y \circ w)/g''(y \circ w)$ of two forms of the same degree in $y \circ w$ with coefficients in R, with the condition $g''(\eta \circ \bar{w}(M)) \neq 0$. This shows that

$$\mathfrak{L}(w(\mathfrak{L}(y(R)/\eta))/\bar{w}(M))$$

is contained in $\mathfrak{L}(y \circ w(R)/\eta \circ \bar{w}(M)) = \mathfrak{L}(z(R)/\bar{z}(M))$. In a similar way we can show that conversely $\mathfrak{L}(z(R)/\bar{z}(M))$ is also contained in

$$\mathfrak{L}(w(\mathfrak{L}(y(R)/\eta))/\bar{w}(M)).$$

If the point η is simple in $y(R)$, then the local ring $\mathfrak{L}(y(R)/\eta)$ is regular; it follows then from what we have said before about the quadratic transform of a regular local ring that (a) $\bar{w}(\mathfrak{L}(y(R)/\eta)) = \bar{S}^{\tau-1}$, (b) $w(\mathfrak{L}(y(R)/\eta))$ is simple at every point $\bar{w}(M)$ in $\bar{w}(\mathfrak{L}(y(R)/\eta))$, and (c) if τ' is the dimension of the local ring $\mathfrak{L}(w(\mathfrak{L}(y(R)/\eta))/\bar{w}(M))$, then $\tau' + \dim(\bar{w}(M)/\bar{R}(\eta)) = \tau$. In view of Lemma 3.1 and the remarks immediately preceding it, we can conclude in this case that (a) $T(\eta)$ consists of a single $(\tau-1)$-dimensional variety, defined over $\bar{R}(\eta)$, and there is a biregular birational transformation Ψ, defined over $\bar{R}(\eta)$, which maps $\bar{w}(\mathfrak{L}(y(R)/\eta)) = \bar{S}^{\tau-1}$ onto $T(\eta)$; (b) $z(R)$ is simple at every point $\bar{z}(M)$ in $T(\eta)$; and (c) τ' is also the dimension of the local ring $\mathfrak{L}(z(R)/\bar{z}(M))$ and we have the relation $\tau' + \dim(\bar{z}(M)/\bar{R}(\eta)) = \tau$.

We shall now assume further that the point η (already assumed to be simple in $y(R)$) is contained in a variety Ω in $\bar{y}(R)$ with the following properties: (1) Ω is defined over an extension field Δ of $\bar{R}$ such that if $\sigma-1$ is the dimension of Ω, then we have the equation $\sigma + \dim(\Delta/\bar{R}) = t$; (2) if ξ is any point in Ω, then $\bar{R}(\xi)$ contains Δ; (3) every point in Ω is simple in $y(R)$; (4) there is a biregular transformation Φ, defined over Δ, which maps the projective space $\bar{S}^{\sigma-1}$ onto Ω. Let ζ be a generic point of Ω over Δ, and let ζ' be a generic point of $T(\eta)$ over $\bar{R}(\eta)$; we shall prove the following lemma, which plays an essential part in the proof of Theorem 1 later.

LEMMA 3.2. *There exists a linearly connected set* Γ *in* $\bar{z}(R)$ *with the properties:* (1) Γ *contains both* $T(\zeta)$ *and* ζ', *and* (2) *every point in* Γ *is simple in* $z(R)$ *and has a dimension at least* $t-2$ *over* $\bar{R}$, *and* (3) Γ *consists of a finite number of rational curves.*

Proof. Since the point η is a specialization of the point ζ over $\bar{R}$, in fact over Δ, the local ring $\mathfrak{L}(y(R)/\zeta)$ is the quotient ring of the local ring $\mathfrak{L}(y(R)/\eta)$ with respect to a prime ideal $\mathfrak{p}_0$, which must then necessarily be of rank 1 (for $\mathfrak{L}(y(R)/\zeta)$ has the dimension 1). The residue ring $\mathfrak{L}(y(R)/\eta)/\mathfrak{p}_0$ is then canonically isomorphic to a subring in the residue field $\bar{R}(\zeta) = \Delta(\zeta)$ of $\mathfrak{L}(y(R)/\zeta)$; in fact, it is easily seen that this subring is precisely the specialization ring $\mathfrak{L}(\zeta/\eta, \Delta)$ in $\Delta(\zeta)$ of the variety Ω at the point η. For the sake of convenience, we shall identify the residue ring $\mathfrak{L}(y(R)/\eta)/\mathfrak{p}_0$ with $\mathfrak{L}(\zeta/\eta, \Delta)$; this implies in particular that we shall identify the residue field of the local ring $\mathfrak{L}(y(R)/\eta)/\mathfrak{p}_0$ with $\bar{R}(\eta) = \Delta(\eta)$, and in this way attach $\mathfrak{L}(y(R)/\eta)/\mathfrak{p}_0$ to Λ. Since Ω is biregularly equivalent over Δ to the projective space $\bar{S}^{\sigma-1}$, and since it is well-known (see [11], Lemma 9, p. 541) that the specialization ring over Δ of $\bar{S}^{\sigma-1}$ at any point is a

regular local ring whose dimension is equal to $\sigma - 1$ minus the dimension of the point over Δ, it follows that $\mathfrak{Q}(y(R)/\eta)/\mathfrak{p}_0$ is a regular local ring of dimension $\sigma - 1 - \dim(\eta/\Delta)$. Since $\mathfrak{Q}(y(R)/\eta)$ is itself a regular local ring, it follows from a well-known result ([3], § III, Proposition 9, p. 705) that the basis $w_1, \cdots, w_\tau$ of $\mathfrak{Q}(y(R)/\eta)$ can be so chosen that $\mathfrak{p}_0$ is the principal ideal (w_τ) generated by the one element w_τ, and that the canonical images $\omega_1, \cdots, \omega_{\tau-1}$ of $w_1, \cdots, w_{\tau-1}$ in $\mathfrak{Q}(y(R)/\eta)/\mathfrak{p}_0$ coincide with any given minimal basis for its maximal prime ideals; furthermore, we have evidently the relation $\dim(\eta/\Delta) = \sigma - \tau$, and since $\bar{R}(\eta)$ contains Δ, it follows that $\dim(\eta/\bar{R}) = \dim(\eta/\Delta) + \dim(\Delta/\bar{R}) = t - \tau$. Now, let $\mathbf{M}$ be the quadratic valuation ring of the regular local ring $\mathfrak{Q}(y(R)/\eta)/\mathfrak{p}_0$; we shall attach $\mathbf{M}$ to Λ in such a way that its residue field $\mathbf{M}$ contains $\bar{R}(\eta)$ and is free with respect to $\bar{R}(\zeta')$ over $\bar{R}(\eta)$. If we denote by ω the point in $S^{\tau-1}$ (observe that by our convention $\bar{S}^{\tau-1}$ is contained in $S^{\tau-1}$) with the projective coordinates $(\omega_1, \cdots, \omega_{\tau-1}, 0)$, then the point $\bar{\omega}(\mathbf{M})$ in $\bar{S}^{\tau-1}$ has the form $(\lambda_1, \cdots, \lambda_{\tau-1}, 0)$, where $\lambda_1 = 1$, and $\lambda_2, \cdots, \lambda_{\tau-1}$ are independent variable (in Λ) over $\Delta(\eta)$. Let M be the valuation ring of the composite valuation obtained from the valuation of $\mathfrak{Q}(y(R)/\zeta)$ and the valuation of its residue field determined by $\mathbf{M}$, i.e. the ring M consists of all elements in $\mathfrak{Q}(y(R)/\zeta)$ whose images in the residue field are contained in the ring $\mathbf{M}$; we shall attach M to Λ by identifying its residue field with the residue field of $\mathbf{M}$. It is clear that M dominates $\mathfrak{Q}(y(R)/\eta)$, so that $\bar{z}(M)$ is a point in $T(\eta)$. Furthermore, it follows from the definition of a composite valuation that $\bar{w}(M) = \bar{\omega}(\mathbf{M}) = (\lambda_1, \cdots, \lambda_{\tau-1}, 0)$; since $\bar{R}(\eta, \bar{z}(M)) = \bar{R}(\eta, \bar{w}(M))$, the point $\bar{z}(M)$ has the dimension $\tau - 2$ over $\bar{R}(\eta)$, and since $\bar{R}(\bar{z}(M))$ contains $\bar{R}(\eta)$, by the relation $\eta = T^{-1}(\bar{z}(M))$, the point $\bar{z}(M)$ has the dimension $\tau - 2 + \dim(\eta/\bar{R}) = t - 2$ over $\bar{R}$. If we denote by Γ' the line in $\bar{S}^{\tau-1}$ joining the points $\bar{w}(M)$ and $\Psi^{-1}(\zeta')$, then it is easily seen that every point in Γ' has a dimension at least $\tau - 2$ over $\bar{R}(\eta)$; going over to the variety $T(\eta)$, we obtain a rational curve $\Psi(\Gamma')$ in $T(\eta)$ which contains both $\bar{z}(M)$ and ζ', such that every point on it has a dimension at least $\tau - 2$ over $\bar{R}(\eta)$. Since every point in $\Psi(\Gamma')$ has the point η as its image under the holomorphic rational mapping T^{-1} (of $z(R)$) and since $\dim(\eta/\bar{R}) = t - \tau$, it follows that every point in $\Psi(\Gamma')$ has a dimension at least $t - 2$ over $\bar{R}$; furthermore, every point in $\Psi(\Gamma')$, being a point in $T(\eta)$, is simple in $z(R)$. This shows that the curve $\Gamma_1 = \Psi(\Gamma')$ has the property (2) in our lemma.

We set $\theta = (\theta_1, \cdots, \theta_\sigma) = \Phi^{-1}(\zeta)$ and $\alpha = (\alpha_1, \cdots, \alpha_\sigma) = \Phi^{-1}(\eta)$; we observe that $\mathfrak{Q}(\zeta/\eta, \Delta)$ is also the specialization ring $\mathfrak{Q}(\theta/\alpha, \Delta)$ in $\Delta(\theta) = \Delta(\zeta)$ of $\bar{S}^{\sigma-1}$ at the point α, so that the elements $\omega_1, \cdots, \omega_{\tau-1}$ form also a basis for

the maximal prime ideal in $\mathfrak{Q}(\theta/\alpha, \Delta)$. We shall show below that there exists a rational curve Γ'' in $\bar{S}^{\sigma-1}$ containing the points θ and α, with the property that every point in Γ'' except possibly α has a dimension at least $\sigma-2$ over Δ and that the elements ω_i/ω_1, $i=2, \cdots, \tau-1$, considered as function on Γ'', have values at α which are independent variables over $\Delta(\alpha)$. Going over to the variety Ω, we have a rational curve $\Phi(\Gamma'')$ containing the points ζ and η such that every point in $\Phi(\Gamma'')$ except possibly η has a dimension at least $\sigma-2$ over Δ and hence a dimension at least $t-2$ over $\bar{R}$. Since the values of ω_i/ω_1, $i=2, \cdots, \tau-1$, on $\Phi(\Gamma'')$ at the point η are independent variables over $\Delta(\alpha)=\bar{R}(\eta)$, we can take them to be the elements $\lambda_2, \cdots, \lambda_{\tau-1}$ introduced before; then in the "proper transform" of $\Phi(\Gamma'')$ in $\overline{y \circ w}(R)$, i.e. the locus over the defining field of $\Phi(\Gamma'')$ of the image in $\overline{y \circ w}(R)$ of a generic point of $\Phi(\Gamma'')$, the (unque) point corresponding to η is $\eta \circ \bar{w}(M)$. Let Γ_2 be the "proper transform" of $\Phi(\Gamma'')$ in $\bar{z}(R)$; it is clear that Γ_2 contains the point $T(\zeta)$. Since the correspondence T is biregular at every point in $\Phi(\Gamma'')$ except η, it follows that every point in Γ_2 outside of $T(\eta)$ has a dimension at least $t-2$ over $\bar{R}$; on the other hand, since the correspondence between $z(R)$ and $y \circ w(R)$ is biregular at every point in $T(\eta)$, it follows that $\bar{z}(M)$ is the (unique) point on Γ_2 corresponding to η in $\Phi(\Gamma'')$. This shows that Γ_2 contains the points $\bar{z}(M)$ and $T(\zeta)$, and has the property (2) in our lemma. If we set $\Gamma=\Gamma_1+\Gamma_2$, then Γ is clearly linearly connected and has the properties (1), (2) and (3) in our lemma.

It remains therefore to prove the existence of the curve Γ''. For convenience we shall assume that ζ is a generic point of Ω over $\bar{R}(\eta)$, not only over Δ; this does not essentially affect our situation, for if Γ_2 does not contain ζ, but contains another generic point χ of Ω over Δ, then we can take a line Γ''' in $\bar{S}^{\sigma-1}$ joining θ and $\Phi^{-1}(\chi)$ and replace Γ by $\Gamma+T(\Phi(\Gamma'''))$. Also, without any loss of generality, we can assume that $\theta_\sigma=\alpha_\sigma=1$ and that $\alpha_\tau, \cdots, \alpha_{\sigma-1}$ are independent variables over Δ; we observe that since θ is a generic point of $\bar{S}^{\sigma-1}$ over $\Delta(\alpha)$, the elements $\theta_1, \cdots, \theta_{\sigma-1}$ are independent variables over $\Delta(\alpha)$. Finally, we recall that we can choose the basis $\omega_1, \cdots, \omega_{\tau-1}$ in any way to suit our convenience; besides, the choice of this basis does not matter any way in our present context, for any two such bases can be obtained one from the other by a linear transformation with coefficients in $\mathfrak{Q}(\theta/\alpha, \Delta)$ and with determinant which is a unit.

For $i=1, \cdots, \tau-1$, let ω_i be an element in $\Delta[\theta_1, \cdots, \theta_i, \theta_\tau, \cdots, \theta_{\sigma-1}]$ which on substituting $\theta_1=\alpha_1, \cdots, \theta_{i-1}=\alpha_{i-1}$, $\theta_\tau=\alpha_\tau, \cdots, \theta_{\sigma-1}=\alpha_{\sigma-1}$ becomes an irreducible polynomial in $\Delta(\alpha_1, \cdots, \alpha_{i-1}, \alpha_\tau, \cdots, \alpha_{\sigma-1})[\theta_i]$ having α_i as a root. It can be easily shown by induction on the dimension $\tau-1$ that

the elements $\omega_1, \cdots, \omega_{\tau-1}$ form a minimal basis for the maximal prime ideal in $\mathfrak{Q}(\theta/\alpha, \Delta)$, this being in fact the proof in [11], Lemma 9. If we denote by s_i the inseparability index of α_i over $\Delta(\alpha_1, \cdots, \alpha_{i-1}, \alpha_\tau, \cdots, \alpha_{\sigma-1})$, then we can evidently choose ω_i so that it involves only the s_i-th power of θ_i. Now, by a reordering of the first $\tau-1$ coordinates in $\bar{S}^{\sigma-1}$ if necessary, we can evidently obtain that the sequence of integers $s_1, \cdots, s_{\tau-1}$ has the property that if $s'_1, \cdots, s'_{\tau-1}$ is another such sequence corresponding to any other ordering of the first $\tau-1$ coordinates in $\bar{S}^{\sigma-1}$, then the last non-vanishing number in the sequence $s'_1 - s_1, \cdots, s'_{\tau-1} - s_{\tau-1}$ is positive. Then, for $j < i \leqq \tau - 1$, the inseparability index of α_j over $\Delta(\alpha_1, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_i, \alpha_\tau, \cdots, \alpha_{\sigma-1})$ cannot be less than s_i; for, if this is not true for one such pair of indices i and j, and if we interchange the i-th and the j-th coordinates, then we would obtain a new sequence $s'_1, \cdots, s'_{\tau-1}$ such that $s_{i+1} = s'_{i+1}, \cdots, s_{\tau-1} = s'_{\tau-1}$, and $s'_i < s_i$, in contradiction to our choice of the sequence $s_1, \cdots, s_{\tau-1}$. Consider now ω_i as a polynomial in one variable θ_j $(j < i)$ with coefficients in $\Delta[\theta_1, \cdots, \theta_{j-1}, \theta_{j+1}, \cdots, \theta_i, \theta_\tau, \cdots, \theta_{\sigma-1}]$; it is clear that in our choice of ω_i we can delete any term whose coefficient vanishes for $\theta = \alpha$. If we delete all such terms from ω_i, then the resulting ω_i will involve only the s_i-th powers of θ_j; for, otherwise, if we set $\theta = \alpha$ in every coefficient in ω_i (considered as a polynomial in θ_j), we would obtain a polynomial in θ_j with coefficients in $\Delta(\alpha_1, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_i, \alpha_\tau, \cdots, \alpha_{\sigma-1})$ which has α_j as a root and which is not a polynomial in the s_i-th power of θ_j, in contradiction to the fact that the inseparability index of α_j over $\Delta(\alpha_1, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_\tau, \cdots, \alpha_{\sigma-1})$ cannot be less than s_i. If we do this for every $j < i$, we would then obtain that ω_i involves only the s_i-th powers of the elements $\theta_1, \cdots, \theta_i$. Thus we have shown that the elements $\omega_1, \cdots, \omega_{\tau-1}$ can be so chosen that each ω_i on substituting $\theta_\tau = \alpha_\tau, \cdots, \theta_{\sigma-1} = \alpha_{\sigma-1}$ becomes a polynomial in the s_i-th powers of $\theta_1, \cdots, \theta_i$; this implies that over an algebraic (purely inseparable) extension Δ' of $\Delta(\alpha)$, each ω_i will have the form $\omega_i = \phi_i(\theta_1 - \alpha_1, \cdots, \theta_i - \alpha_i)^{s_i} + \rho_i$, where ϕ_i is a ploynomial with coefficients in Δ' and ρ_i is an element in $\Delta'[\theta_1, \cdots, \theta_{\sigma-1}]$ which vanishes for $\theta_\tau = \alpha_\tau, \cdots, \theta_{\sigma-1} = \alpha_{\sigma-1}$. Furthermore, it is clear that $\phi_i(0, \cdots, 0) = 0$ and that the coefficient β_i of the term $(\theta_i - \alpha_i)^{s_i}$ in ϕ_i does not vanish; it follows that there exists a power series $\psi_i(\theta_1 - \alpha_1, \cdots, \theta_{i-1} - \alpha_{i-1})$ in $\theta_1 = \alpha_1, \cdots, \theta_{i-1} - \alpha_{i-1}$ with coefficients in Δ' such that $\phi_i(\theta_1 - \alpha_1, \cdots, \theta_{i-1} - \alpha_{i-1}, \psi_i) = 0$.

Let $\mu_1, \cdots, \mu_{\sigma-1}$ be independent variables over $\Delta'(\theta)$; for convenience in notation, we set $s_i = 1$ and $\psi_i = 0$ for $i = \tau, \cdots, \sigma-1$. We define a curve Γ'' in $\bar{S}^{\sigma-1}$ by the following recursive parametric equations, where $Y_1, \cdots, Y_{\sigma-1}$

denote the (affine) coordinates in $\bar{S}^{\sigma-1}$ (with $Y_\sigma = 1$) and T denotes the parameter:

$$Y_i = \alpha_i + \mu_i T^{(s/s_i)} - \psi_i(Y_1 - \alpha_1, \cdots, Y_{i-1} - \alpha_{i-1})_s + \nu_i T^{s+1},$$

for $i = 1, \cdots, \sigma - 1$, where the $Y_1, \cdots, Y_{i-1}$ on the right hand side are to be substituted by their expressions in the previous equations, and the subscript s after the ψ_i indicates that all terms of degrees higher than s are to be deleted; furthermore, each ν_i is to be determined by the condition $\nu_i = \theta_i - \alpha_i - \mu_i + [\psi_i(Y_1 - \alpha_1, \cdots, Y_{i-1} - \alpha_{i-1})_s]_{T=1}$. It is easily seen that Γ'' is a rational curve and contains the points α (for $T = 0$) and θ (for $T = 1$). Furthermore, if $\Gamma''(\gamma)$ is the point on Γ'' corresponding to any value $\gamma \neq 0$ of T, then $\Delta'(\mu, \nu)$ is contained in $\Delta'(\gamma, \Gamma''(\gamma), \nu)$, and since $\Delta'(\mu, \nu)$ has the dimension $\sigma - 1$ over $\Delta'(\nu)$, it follows that $\Gamma''(\gamma)$ must have a dimension at least $\sigma - 2$ over $\Delta'(\nu)$. For $T = \infty$, the corresponding point on Γ'' has the homogeneous coordinates $(\nu_1, \cdots, \nu_{\sigma-1}, 0)$ and hence has the dimension $\sigma - 2$ over $\Delta'(\mu)$. Thus every point on Γ'', except possibly the point α, has a dimension at least $\sigma - 2$ over Δ. Finally, a simple calculation shows that each ω_i, considered as a function on Γ'', has at the point α a power series development in T beginning with the term $(\beta_i \mu_i + \delta_i) T^s$, where δ_i is an element in $\Delta'(\theta_1, \cdots, \theta_{i-1})$; this shows that the ratios ω_i/ω_1, $i = 2, \cdots, \tau - 1$, considered as functions on Γ'', have values at α which are independent variables over $\Delta(\alpha)$. This concludes the proof of Lemma 3.2.

Remark. If $\sigma = 2$, then Ω is a rational curve already and the rather complicated construction of the curve Γ'' is not necessary. In this case, the curve Ω can contain only a finite number of fundamental points $\eta^{(1)}, \cdots, \eta^{(d)}$ for T, and each set $T(\eta^{(i)})$ is also a rational curve with the property (2). It is clear that the set $T(\Omega)$ consists of the "proper transform" of Ω and all the curves $T(\eta^{(1)}), \cdots, T(\eta^{(d)})$, and that it is a connected set with the properties (2) and (3).

Let M be a valuation ring in K dominating R. We start with a rational transform $y(R)$ of R, and obtain from it a sequence of rational transforms $z^{(i)}(R)$ of R, $i = 1, 2, \cdots$, such that each $z^{(i)}(R)$ is the quadratic transform of $z^{(i-1)}(R)$ centered at $\bar{z}^{(i-1)}(M)$, whereby we set $z^{(0)}(R) = y(R)$. We shall say that $z^{(i)}(R)$ is the *i-th quadratic transform of* $y(R)$ *along* M. As we have observed above, if $y(M)$ is simple in $y(R)$, then $\bar{z}^{(i)}(M)$ will also be simple in $z^{(i)}(R)$ for every i; and, in this case, if we denote by t_i the dimension of $\mathfrak{Q}(z^{(i)}(R)/\bar{z}^{(i)}(M))$, then we have the relation $t_i + \dim(\bar{z}^{(i)}(M)/\bar{R}(\bar{z}^{(i-1)}(M))) = t_{i-1}$, for every $i = 1, 2, \cdots$, while for $i = 0$ we have the relation

$t_0 + \dim(\bar{z}^{(0)}(M)/\bar{R}) = t$. Since the birational correspondence between $z^{(i-1)}(R)$ and $z^{(i)}(R)$ is everywhere regular in $z^{(i)}(R)$, it follows that $\bar{R}(z^{(i)}(M))$ contains $\bar{R}(\bar{z}^{(i-1)}(M))$; we obtain therefore by addition the relation $t_i + \dim(\bar{z}^{(i)}(M)/\bar{R}) = t$ for every $i = 0, 1, \cdots$. We shall say that the sequence $z^{(i)}(R)$ of quadratic transforms of $y(R)$ along M is *finite* if there exists an index i such that $t_i = 1$ or equivalently $\dim(\bar{z}^{(i)}(M)/\bar{R}) = t - 1$. It is clear that this is possible only when the residue field of M has a dimension at least $t-1$ over $\bar{R}$; according to [2], Theorem 1, this implies that M is a real discrete valuation ring and that its residue field has exactly the dimension $t-1$ over $\bar{R}$. It is important for our present purpose that the converse of this statement is also true.

LEMMA 3.3. *If M is a real discrete valuation ring in $\mathfrak{V}(R)$ such that its residue field has the dimension $t-1$ over $\bar{R}$, and if $y(R)$ is a rational transform of R such that $y(M)$ is simple in $y(R)$, then the sequence of quadratic transforms of $y(R)$ along M is finite.*

A proof of this lemma is given in [2], Proposition 3.

We are now ready to prove the Linear Connectedness Theorem.

THEOREM 1 (Linear Connectedness Theorem). *The center $\bar{y}(R)$ of a rational transform $y(R)$ of a regular local ring R is linearly connected.*

Proof. As before, we denote by t, K, and $\bar{R}$ the dimension, the quotient field, and the residue field respectively of R. Consider the quadratic transform $z^{(1)}(R)$ of R and let T_1 be the birational correspondence between $z^{(1)}(R)$ and $y(R)$; if $\zeta^{(1)}$ is a generic point of $\bar{z}^{(1)}(R)$ over $\bar{R}$ (observe that $\bar{z}^{(1)}(R) = \bar{S}^{t-1}$), then T_1 is regular at $\zeta^{(1)}$ and hence $T_1(\zeta^{(1)})$ is a uniquely determined point in $\bar{y}(R)$, which we shall denote by ζ. To prove our theorem, it is sufficient to show that any point η in $\bar{y}(R)$ is linearly connected in $\bar{y}(R)$ to this point ζ. By Lemma 2.7, there exists a real discrete valuation ring M in $\mathfrak{V}(R)$, with a residue field which has the dimension $t-1$ over $\bar{R}$, such that $\bar{y}(M) = \eta$. For $1 < i \leqq a$ (the integer a will be determined presently), let $z^{(i)}(R)$ be the $(i-1)$-th quadratic transform of $z^{(1)}(R)$ along M, and let T_i be the birational correspondence between $z^{(i)}(R)$ and $y(R)$. We set $T_{i,j} = T_j^{-1}T_i$ for $i, j = 1, \cdots, a$, so that $T_{i,j}$ is the birational correspondence between $z^{(i)}(R)$ and $z^{(j)}(R)$; we set $\eta^{(i)} = \bar{z}^{(i)}(M)$ for $i = 1, \cdots, a$, and take for each $i = 2, \cdots, a$, a generic point $\zeta^{(i)}$ of the variety $T_{i-1,i}(\eta^{(i-1)})$ over $\bar{R}(\eta^{(i-1)})$. The integer a is to be taken as the smallest integer such that the point $\eta^{(a)} = \bar{z}^{(a)}(M)$ has the dimension $t-1$ over $\bar{R}$; by Lemma 3.3, such an integer a exists, and since $\eta^{(a)}$ is simple in $z^{(a)}(R)$, it must be a regular

point of T_a and we have evidently $\eta = T_a(\eta^{(a)})$. Furthermore, since $\eta^{(a)}$ has the dimension $t-1$ over $\bar{R}$, it must have the maximum possible dimension over $\bar{R}(\eta^{(a-1)})$ and hence is a generic point of $T_{a-1,a}(\eta^{(a-1)})$ over $\bar{R}(\eta^{(a-1)})$; we can therefore take $\eta^{(a)}$ itself as the point $\zeta^{(a)}$, which will be convenient for us later. We maintain now that there exists a connected set $\Gamma^{(i)}$ in $\bar{z}^{(i)}(R)$ with the following properties: (1) $\Gamma^{(i)}$ contains both $\zeta^{(i)}$ and $T_{1,i}(\zeta^{(1)})$, (2) every point in $\Gamma^{(i)}$ is simple in $z^{(i)}(R)$ and has a dimension at least $t-2$ over $\bar{R}$, and (3) $\Gamma^{(i)}$ consists of either a single point or a finite number of rational curves. We shall show this by induction on the index i, beginning by setting $\Gamma^{(1)} = \zeta^{(1)}$, and, assuming the induction hypothesis that the set $\Gamma^{(i-1)}$ has already been constructed, we shall proceed to contruct the set $\Gamma^{(i)}$. If $T_{i-1,i}$ has no fundamental points in $\Gamma^{(i-1)}$ and hence is biregular everywhere in $\Gamma^{(i-1)}$, then the set $T_{i-1,i}(\Gamma^{(i-1)})$ is evidently connected, has the properties (2) and (3), and contains the points $T_{1,i}(\zeta^{(1)}) = T_{i-1,i}(T_{1,i-1}(\zeta^{(1)}))$ and $T_{i-1,i}(\zeta^{(i-1)})$. Assuming now that $T_{i-1,i}$ has some fundamental points in $\Gamma^{(i-1)}$; we observe that this can occur only if $\dim(\eta^{(i-1)}/\bar{R}) = t-2$, in which case $T_{i-1,i}$ can have only a finite number of fundamental points $\eta^{(i-1,1)}, \cdots, \eta^{(i-1,b)}$ in $\Gamma^{(i-1)}$, all of dimension $t-2$ over $\bar{R}^{(i-1)}$, and each set $T_{i-1,i}(\eta^{(i-1,j)})$ is a rational curve in $\bar{z}^{(i)}(R)$ which has the property (2). If we denote by $\Gamma_1^{(i-1)}, \cdots, \Gamma_c^{(i-1)}$ the distinct component curves in $\Gamma^{(i-1)}$, then by the Remark after Lemma 3.2 each set $T_{i-1,i}(\Gamma_k^{(i-1)})$ is connected and has the properties (2) and (3), and it will contain the curve $T_{i-1,i}(\eta^{(i-1,j)})$ whenever $\Gamma_k^{(i-1)}$ contains the point $\eta^{(i-1,j)}$. It follows then from an elementary combinatorial consideration that the set $T_{i-1,i}(\Gamma^{(i-1)}) = \sum_{k=1}^{c} T_{i-1,i}(\Gamma_k^{(i-1)})$ is connected, and it evidently has the properties (2) and (3), and contains the point $T_{1,i}(\zeta^{(1)})$ and $T_{i-1,i}(\zeta^{(i-1)})$. Now, by Lemma 3.2, there exists a connected set Γ_i in $\bar{z}^{(i)}(R)$ which has the properties (2) and (3) and contains the points $\zeta^{(i)}$ and $T_{i-1,i}(\zeta^{(i-1)})$. The set $\Gamma^{(i)} = T_{i-1,i}(\Gamma^{(i-1)}) + \Gamma_i$ is then evidently connected and has the properties (1), (2), and (3).

For $i = a$, we obtain a linearly connected set $\Gamma^{(a)}$ in $\bar{z}^{(a)}(R)$ which has the properties (2) and (3) and which contains the points $T_{1,a}(\zeta^{(1)})$ and $\bar{z}^{(a)}(M) = \eta^{(a)} = \zeta^{(a)}$. If T_a is regular at every point in $\Gamma^{(a)}$, then T_a induces a continuous mapping of $\Gamma^{(a)}$ into $\bar{y}(R)$, so that $T_a(\Gamma^{(a)})$ is a linearly connected set; since $T_a(\Gamma^{(a)})$ contains the points $\zeta = T_1(\zeta^{(1)}) = T_a(T_{1,a}(\zeta^{(1)}))$ and $\eta = T_a(\eta^{(a)}) = T_a(\zeta^{(a)})$, our theorem would therefore be proved. Otherwise, we proceed as follows. Consider the join $y \circ z^{(a)}(R)$, and let T_a' be the birational correspondence between $z^{(a)}$ and $y \circ z^{(a)}(R)$. By Lemma 2.5, there exist in $\overline{y \circ z^{(a)}}(R)$ only a finite number of points of dimension $t-1$ over $\bar{R}$

such that no two of them are equivalent over $\bar{R}$ (i.e. specializations of each other over $\bar{R}$), and by Lemma 2.6, there exist only a finite number of valuation rings in $\mathfrak{V}(R)$ whose centers in $y \circ z^{(a)}(R)$ coincide with any one of these points, and these valuation rings are all real discrete. There exist therefore a finite number of real discrete valuation rings $M_1, \cdots, M_d$ in $\mathfrak{V}(R)$ such that any valuation ring in $\mathfrak{V}(R)$ whose center in $y \circ z^{(a)}(R)$ has the dimension $t-1$ over $\bar{R}$ coincides with one of them up to attachments to Λ. If T_a is not regular at a point ξ in $\Gamma^{(a)}$, then T_a' also cannot be regular at ξ; since ξ is simple and hence normal in $z^{(a)}(R)$, it follows from Lemma 2.4 that $T_a'(\xi)$ has at least one component of positive dimension. There exists therefore a valuation ring N in $\mathfrak{V}(R)$ such that $\bar{z}^{(a)}(N) = \xi$ and such that the point $\overline{y \circ z^{(a)}}(N)$ has a positive dimension over $\bar{R}(\xi)$. Since $T_a'^{-1}$ is regular everywhere, $\bar{R}(\overline{y \circ z^{(a)}}(N))$ must contain $\bar{R}(\xi)$ as subfield, and since ξ has the dimension $t-2$ over $\bar{R}$, it follows that $\overline{y \circ z^{(a)}}(N)$ has the dimension $t-1$ over $\bar{R}$. From what we have just said, N must coincide up to attachment to Λ with one of the valuation rings M_i, say M_1, so that $z^{(a)}(M_1)$ is equivalent to ξ over $\bar{R}$ and hence is also simple in $z^{(a)}(R)$. Now, let $z^{(i)}(R)$, for $a < i \leqq a_1$ (the integer a_1 will be determined presently), be the $(i-a)$-th quadratic transform of $z^{(a)}(R)$ along M_1, and let T_i be as before the birational correspondence between $z^{(i)}(R)$ and $y(R)$; also as before, we set $T_{i,j} = T_j^{-1} T_i$, so that $T_{i,j}$ is the birational correspondence between $z^{(i)}(R)$ and $z^{(j)}(R)$. If we set $\Gamma^{(i)} = T_{i-1,i}(\Gamma^{(i-1)}) = T_{a,i}(\Gamma^{(a)})$, then by exactly the same induction argument as before we can show that $\Gamma^{(i)}$ is a linearly connected set with the properties (2) and (3), and it is evident that $\Gamma^{(i)}$ contains the points $T_{1,i}(\zeta^{(1)}) = T_{a,i}(T_{1,a}(\zeta^{(1)}))$ and $\bar{z}^{(i)}(M) = T_{a,i}(\eta^{(a)})$. By Lemma 3.3, there exists a smallest integer $a_1 > a$ such that the point $\bar{z}^{(a_1)}(M_1)$ has the dimension $t-1$ over $\bar{R}$, so that T_{a_1} is regular at $\bar{z}^{(a_1)}(M_1)$; this determines a_1. If T_{a_1} is regular at every point in $\Gamma^{(a_1)}$, then T_{a_1} induces a continuous mapping of $\Gamma^{(a_1)}$ into $\bar{y}(R)$ and hence $T_{a_1}(\Gamma^{(a_1)})$ is a linearly connected set; since $T_{a_1}(\Gamma^{(a_1)})$ contains the points $\zeta = T_1(\zeta^{(1)}) = T_{a_1}(T_{1,a_1}(\zeta^{(1)}))$ and $\eta = T_{a_1}(\bar{z}^{(a_1)}(M))$, this would prove our theorem. If T_{a_1} is not regular at some one point $\xi^{(1)}$ in $\Gamma^{(a_1)}$, then by the same argument as before we can show that there exists a valuation ring N_1 in $\mathfrak{V}(R)$ such that $\bar{z}^{(a_1)}(N_1) = \xi^{(1)}$ and such that the point $\overline{y \circ z}^{(a_1)}(N_1)$ has the dimension $t-1$ over $\bar{R}$. Since $\bar{z}^{(a)}(N_1)$ is a point in $\Gamma^{(a)}$, it must have the same dimension $t-2$ over $\bar{R}$ as the point $\bar{z}^{(a_1)}(N_1)$; it follows that the point $\bar{z}^{(a_1)}(N_1)$ is algebraic over $\bar{R}(\bar{z}^{(a)}(N_1))$, and hence the point $\overline{y \circ z}^{(a_1)}(N_1)$ must be algebraic over $\bar{R}(\overline{y \circ z}^{(a)}(N_1))$. This shows that the point $\overline{y \circ z}^{(a)}(N_1)$ has the dimension $t-1$ over $\bar{R}$, and it follows from what we have said before that N_1 must coincide up to attachments to Λ with one

of the valuation rings M_i, say M_2. We can then proceed in a similar way with $z^{(a)}(R)$, $\Gamma^{(a_1)}$, M_2 as we did before with $z^{(a)}(R)$, $\Gamma^{(a)}$ M_1. Thus we proceed in this way step by step until, after a finite number π of steps, we have obtained a rational transform $z^{(\pi)}(R)$ of R with the following properties, whereby T_π and $T_{1,\pi}$ have similar meanings as before: There exists a linearly connected set $\Gamma^{(\pi)}$ in $\bar{z}^{(\pi)}(R)$ which has the properties (2) and (3) and which contains the points $T_{1,\pi}(\zeta^{(1)})$ and $\bar{z}^{(\pi)}(M)$, and for every M_i the point $\bar{z}^{(\pi)}(M_i)$ either has the dimension $t-1$ over $\bar{R}$ or is not equivalent over $\bar{R}$ to a point in $\Gamma^{(\pi)}$. From what we have said above, this implies that T_π is regular at every point in $\Gamma^{(\pi)}$. Therefore T_π induces a continuous mapping of $\Gamma^{(\pi)}$ into $\bar{y}(R)$, so that $T_\pi(\Gamma^{(\pi)})$ is a linearly connected set; since $T_\pi(\Gamma^{(\pi)})$ contains the points $\zeta = T_1(\zeta^{(1)}) = T_\pi(T_{1,\pi}(\zeta^{(1)}))$ and $\eta = T_\pi(\bar{z}^{(\pi)}(M))$, this concludes the proof of Theorem 1.

COROLLARY 1. *In case $t=2$, there exists a finite quadratic transform $z(R)$ of R such that the birational correspondence between $z(R)$ and $y(R)$ is regular at every point in $\bar{z}(R)$.*

Proof. Since in this case every point in $\bar{z}^{(\pi)}(R)$ must have a dimension over $\bar{R}$ which is either $t-1$ or $t-2=0$, it follows from the proof above that T_π is regular at every point in $\bar{z}^{(\pi)}(R)$. We can therefore take $z(R)$ to be $z^{(\pi)}(R)$.

COROLLARY 2. *The topological space $\mathfrak{V}(R)$ is connected.*

This follows immediately from Theorem 1 and the definition of topology in $\mathfrak{V}(R)$ given in section 2.

4. General connectedness theorem and extended principle of Degeneration. We begin by recalling the well-known Hensel's Lemma: If a splitting form $F(X)$ with coefficients in a complete real discrete valuation ring R is irreducible over R (or over the quotient field K of R, which in this case amounts to the same thing), then the residue form $\bar{F}(X)$, if it does not vanish identically, is not the product of two relatively prime forms in $\bar{R}[X]$. Geometrically, this means that if a positive 0-cycle Z is rational and irreducible over K, then its specialization $\bar{Z}(R)$ at R is $\bar{R}$-connected. Our purpose here is to generalize this Hensel's Lemma to the case where R is an arbitrary complete local domain. In a sense something of this nature has already been done in the literature; in fact, it is well-known that the Hensel's Lemma, at least in the case of a form of two variables, holds for any complete local ring R (see e.g. [6], Theorem 4), for the usual proof of the Hensel's Lemma

for a valuation ring can be carried over without any difficulties to the general case. However, this generalization is essentially a trivial one and in reality deals only with a highly specialized situation; for the condition that the residue form $\bar{F}(X)$ does not vanish identically, while easily obtainable in case of a valuation ring by a suitable choice of the factor of proportionality, imposes a very severe restriction on the form $F(X)$ in the general case. Expressed in geometrical terms, this condition means that the cycle Z must have a uniquely determined specialization at R, so that the set $\bar{Z}(R)$ consists of only a single cycle; while this is always true in case of a valuation ring, it evidently represents a severe restriction in the general case. In fact, the fact that $\bar{Z}(R)$ is in general an algebraic set of cycles instead of a single cycle, can be considered as the principal new feature in local algebraic geometry which one encounters in going over from valuation rings to arbitrary local domains. A true generalization of the Hensel's Lemma should therefore take account of this important point, and in lieu of the $\bar{R}$-connectedness of a single residue cycle, it should supply us with some information about the $\bar{R}$-connectedness properties of the entire set of cycles $\bar{Z}(R)$. The General Connectedness Theorem, as formulated in § 1, is a theorem of just such a nature and can therefore very appropriately be considered as a generalization of the Hensel's Lemma, or rather, since it deals with cycles of arbitrary dimensions, as a generalization of the Principle of Degeneration as formulated and proved in [4], which is itself a generalization of the Hensel's Lemma to cycles of higher dimensions while retaining the assumption of a valuation ring. However, it is possible to formulate a theorem which is an even truer generalization of the Principle of Degeneration in the sense that it deals with the $\bar{R}$-connectedness properties not of the point set $|\bar{Z}(R)|$, but rather of the individual cycles in $\bar{Z}(R)$. This leads us to what we shall call the Extended Principle of Degeneration, from which one can easily obtain the General Connectedness Theorem as a simple corollary, as we have already mentioned in § 1.

We shall speak of a *continuous system* of positive r-cycles in $\bar{S}^n$ *parametrized* by a local domain R, if there is a continuous mapping $N \to \Xi(N)$ $(N \in \mathfrak{V}(R))$ of the set $\mathfrak{V}(R)$ of all valuation rings in K dominating R into the set $\mathfrak{Z}(\bar{S}^n)$ of all positive r-cycles in $\bar{S}^n$ such that for every N the cycle $\Xi(N)$ is rational over thet residue field $\bar{N}$ of N. We shall denote such a continuous system of cycles by the symbols $\Xi(R)$. The continuity condition here in the definition refers to the Zariski $\bar{R}$-topologies of both $\mathfrak{V}(R)$ and $\mathfrak{Z}(\bar{S}^n)$; we recall that the topology of $\mathfrak{V}(R)$ is defined by means of the system of Zariski $\bar{R}$-topologies of the centers of all rational transforms of R, as described after Lemma 2.2 in section 2, while the topology of $\mathfrak{Z}(\bar{S}^n)$ is that of the

(discrete) union of an infinite sequence of spaces, each one of which is the set of all positive r-cycles of a given degree in $\bar{S}^n$ and is being endowed with its Zariski $\bar{R}$-topology by means of the associated points. In particular, this implies that if N and N' are two elements in $\mathfrak{V}(R)$ such that N' is the quotient ring of N, then $\Xi(N)$ is a specialization of $\Xi(N')$ over $\bar{R}$. In terms of the associated forms, the continuous system of cycles $\Xi(R)$ can be given by a *continuous system* of forms $\Phi_R(U) = \{\Phi_N(U) \mid N \in \mathfrak{V}(R)\}$ such that each $\Phi_N(U)$ is an associated form and is rational over $\bar{N}$, whereby we observe that these forms are determined only up to proportionality factors and that rationality of a form refers to the ratios of the coefficients.

Let R_1 be a local domain dominating R, not necessarily contained in the quotient field K of R; since every valuation ring in the quotient field of R_1 dominating R_1 contracts in K to a valuation ring dominating R, there is defined in this way a canonical mapping of $\mathfrak{V}(R_1)$ into $\mathfrak{V}(R)$, which is easily seen to be continuous. If we set $\Xi(N) = \Xi(K \cap N)$ for every N in $\mathfrak{V}(R_1)$, then the mapping $N \to \Xi(N)$ $(N \in \mathfrak{V}(R_1))$ defines evidently a continuous system parametrized by R_1, which we shall denote by $\Xi(R_1)$ and call the *subsystem* of $\Xi(R)$ determined by R_1.

We remark that in general the cycles in a continuous system $\Xi(R)$ can very well have different degrees; in particular, for some N the cycle $\Xi(N)$ may have the degree 0, i.e. $\Xi(N)$ is the zero r-cycle. If every cycle in a continuous system has the degree 0, so that all of them coincide with the zero r-cycle, then the system is said to be *trivial*. However, in the important case where $\mathfrak{V}(R)$ is connected, as is the case with a regular local ring R according to Theorem 1, Corollary 2, it can be easily seen that all cycles in a continuous system $\Xi(R)$ must have the same degree, so that if for any one N the cycle $\Xi(N)$ has the degree 0, then the system $\Xi(R)$ must be trivial. To see this, one need only to observe that the set of all positive cycles of a given degree d in $\bar{S}^n$ is a closed subset in $\mathfrak{Z}(\bar{S}^n)$ and so is the set of all positive cycles in $\bar{S}^n$ of degrees different from d; the inverse images of these two subsets under the mapping $N \to \Xi(N)$ must then be two disjoint closed subsets in $\mathfrak{V}(R)$ whose union coincides with $\mathfrak{V}(R)$, and hence one of them must be empty. This means that if d is the degree of any one cycle in $\Xi(R)$, then all the other cycles in $\Xi(R)$ must have the same degree d.

If there exist two non-trivial continuous systems of positive r-cycles $\Xi'(R)$ and $\Xi''(R)$ such that we have the equation $\Xi(N) = \Xi'(N) + \Xi''(N)$ for every N in $\mathfrak{V}(R)$, then we shall say that the system $\Xi(R)$ *splits* into the two systems $\Xi'(R)$ and $\Xi''(R)$, or that the pair $(\Xi'(R), \Xi''(R))$ is a splitting of $\Xi(R)$. Such a splitting is said to be $\bar{R}$-disjoint at a valuation ring N if

$\Xi'(N)$ and $\Xi''(N)$ are $\bar{R}$-disjoint, i.e. the supports of $\Xi'(N)$ and $\Xi''(N)$ are disjoint sets in the Zariski $\bar{R}$-topology, and it is said to be *everywhere $\bar{R}$-disjoint* or simply *$\bar{R}$-disjoint* if it is $\bar{R}$-disjoint at every N in $\mathfrak{V}(R)$. In terms of the associated forms, a splitting of $\Xi(R)$ can be given by a splitting of the system of forms $\Phi_R(U)$ into two systems of forms $\Phi'_R(U)$ and $\Phi_R''(U)$, so that for every N in $\mathfrak{V}(R)$ we have the equation $\Phi_N(U) \doteq \Phi'_N(U)\Phi''_N(U)$, where the symbol $\doteq$ denotes equality up to proportionality factors; and we observe that in case $r = 0$, the cycles $\Xi'(N)$ and $\Xi''(N)$ are $\bar{R}$-disjoint if and only if their associated forms $\Phi'_N(U)$ and $\Phi''_N(U)$ are relatively $\bar{R}$-prime, i.e. there does not exist a splitting form in $\bar{R}[U]$ which has common factors with both $\Phi'_N(U)$ and $\Phi''_N(U)$. Finally, a continuous system of cycles is said to be *$\bar{R}$-connected* if it does not have any $\bar{R}$-disjoint splitting.

If Z is a positive r-cycle in S^n, rational over K, then the algebraic set of cycles $\bar{Z}(R)$ forms in a natural way a continuous system parametrized by R, which we shall also denote by the same symbol $\bar{Z}(R)$, there being not much danger of confusion on this score. In this case, we shall say that the cycle Z *generates* the continuous system $\bar{Z}(R)$. If the cycle Z is the sum of two positive r-cycles Z' and Z'', also rational over K, then it is clear that we have the equation $\bar{Z}(N) = \bar{Z}'(N) + \bar{Z}''(N)$ for every N in $\mathfrak{V}(R)$; hence the system $\bar{Z}(R)$ splits into the two systems $\bar{Z}'(R)$ and $\bar{Z}''(R)$. Thus we see that a splitting of the cycle Z induces a splitting of the system $\bar{Z}(R)$ generated by Z. The Extended Principle of Degeneration, which we shall now formulate, asserts that conversely every $\bar{R}$-disjoint splitting of the system $\bar{Z}(R)$ can be obtained in this way. We shall prove this principle here only for the case of a complete regular local ring, although it is very likely that it holds generally for an arbitrary complete local domain.

THEOREM 2 (Extended Principle of Degeneration). *Let $(R, \mathfrak{p})$ be a complete regular local ring, with the quotient field K and the residue field $\bar{R}$, and let Z be a positive cycle in S^n, rational over K; if Z is K-connected, then the continuous system of cycles $\bar{Z}(R)$ is $\bar{R}$-connected.*

The proof of this theorem will be given in the next section. Here we shall show how the General Connectedness Theorem can be obtained from it by a very simple argument.

THEOREM 3 (General Connectedness Theorem). *Let $(R, \mathfrak{p})$ be a complete local domain, with the quotient field K and the residue field $\bar{R}$, and let Z be a positive cycle in S^n, rational over K; if Z is K-connected, then the point set $|\bar{Z}(R)|$ is $\bar{R}$-connected.*

Proof. In case R is regular, this theorem follows almost immediately from Theorem 2. In fact, if there exist two disjoint $\bar{R}$-closed subsets Ω' and Ω'' in $\bar{S}^n$ such that $|\bar{Z}(R)|$ is contained in their union and has non-empty intersections with both of them, and if we denote by $\Xi'(N)$ and $\Xi''(N)$ the restriction of $\bar{Z}(N)$ in Ω' and Ω'' respectively, then we obtain two continuous systems $\Xi'(R) = \{\Xi'(N) \mid N \in \mathfrak{V}(R)\}$ and $\Xi''(R) = \{\Xi''(N) \mid N \in \mathfrak{V}(R)\}$ which are clearly non-trivial, and the equation $\bar{Z}(N) = \Xi'(N) + \Xi''(N)$ defines evidently an $\bar{R}$-disjoint splitting of the system $\bar{Z}(R)$ into the two systems $\Xi'(R)$ and $\Xi''(R)$; it follows then from Theorem 2 that Z cannot be K-connected. It remains to show that the general case of an arbitrary complete local domain can be reduced to this special case.

According to Cohen [6], Theorem 16, there exists a subring R_0 in R such that R_0 is a complete regular local ring having the same dimension and the same residue field as R, and such that R is a finite module over R_0; this implies that K is a finite algebraic extension of the quotient field K_0 of R_0. Let $Z_1 = Z, Z_2, \cdots, Z_s$ be the complete set of conjugates of Z over K_0, and let $K_1 = K, K_2, \cdots, K_s$ and $R_1 = R, R_2, \cdots, R_s$ be the corresponding sets of conjugates of K and R respectively over K_0; let K' be the compositum of all K_i and let R' be the ring generated by all the R_i. Since R' is a finite module over R_0 and since R_0 is complete, it follows from [3], III, Proposition 8, that R' is a complete local domain; since the local rings R_i all have the same residue field $\bar{R}$, we can attach R' to Λ by identifying its residue field also with $\bar{R}$, so that R' dominates all R_i as well as R_0. Furthermore, it is easily seen that for any valuation ring in K_0 dominating R_0, any extension of it in K' must dominate R'; and a similar statement holds also for valuation rings in K_i dominating R_i. We set $Z_0 = \sum_{i=1}^{s} Z_i$, so that Z_0 is rational over K_0; since Z is K-connected, it follows that Z_0 is K_0-connected. In fact, any K_0-irreducible component X_0 in Z_0 is the sum of the complete set of conjugates over K_0 of a K-irreducible component X in Z, and two such components X_0 and Y_0 in Z_0 will certainly meet if the corresponding components X and Y in Z meet. It is sufficient to show that $|\bar{Z}_0(R_0)| = |\bar{Z}(R)|$; for, since R_0 is a complete regular local domain, $|\bar{Z}_0(R_0)|$ must be $\bar{R}$-connected by what we have just shown above, and hence $|\bar{Z}(R)|$ must also be $\bar{R}$-connected.

Since every valuation ring in K_0 dominating R_0 can be extended to a valuation ring in K' dominating R', it follows that $\bar{Z}_0(R_0) = \bar{Z}_0(R')$, and since every valuation ring in K_i dominating R_i can also be extended to a valuation ring in K' dominating R', it follows that $\bar{Z}_i(R_i) = \bar{Z}_i(R')$. It is sufficient to show that $|\bar{Z}(R)| = |\bar{Z}_i(R_i)|$ for every i; for then $|\bar{Z}_0(R_0)|$

$= |\bar{Z}_0(R')| = \bigcup_i |\bar{Z}_i(R')| = \bigcup_i |\bar{Z}_i(R_i)| = |\bar{Z}(R)|$. To prove this we observe first that R_0 contains a coefficient ring k ([6], Theorem 9 and Theorem 11) which is also a coefficient ring for every R_i and is invariant under the isomorphism between R and R_i over R_0. Let a and a' be any two conjugate elements over K_0 in R and R_i respectively, and let b be an element in k such that $\bar{b} = \bar{a}$ (such an element b exists by the definition of a coefficient ring); then $a - b$ is in $\mathfrak{p}$, and hence by isomorphism $a' - b$ is in $\mathfrak{p}_i$ (the maximal prime ideal in R_i), which show that $\bar{b} = \bar{a}'$. Thus any two conjugate elements over K_0 in R and R_i must have the same residue in $\bar{R}$. Let y be the associated point of Z, and let y' be the associated point of Z_i; if $f(Y)$ and $f'(Y)$ are two conjugate forms over K_0 in $R[Y]$ and $R_i[Y]$ respectively, then by isomorphism we have the relation $f(y) = 0$ if and only if $f'(y') = 0$, and it follows from what we have just shown that the residue forms $\bar{f}(Y)$ and $\bar{f}'(Y)$ must coincide. Since $\bar{y}(R)$ and $\bar{y}'(R_i)$ are defined by such residue forms, it follows that the two sets $\bar{y}(R)$ and $\bar{y}'(R_i)$ must coincide; hence the two algebraic sets of cycles $\bar{Z}(R)$ and $\bar{Z}_i(R_i)$ must coincide, and hence $|\bar{Z}(R)| = |\bar{Z}_i(R_i)|$.

We observe here that if we are only concerned with the absolute geometry and assume the absolute connectednesss of Z, then it would be sufficient to invoke the simpler Principle of Degeneration in [4] instead of Theorem 2; only we must then invoke directly the connectedness of $\mathfrak{V}(R)$, which will also be used in the proof of Theorem 2. In fact, in this case Z would be N^*-connected for every N in $\mathfrak{V}(R)$, and therefore by that principle $\bar{Z}(N^*) = \bar{Z}(N)$ would be $\bar{N}$-connected and hence also $\bar{R}$-connected. Then, instead of using Theorem 2, we need only to observe in the first part of the above proof that since $\mathfrak{V}(R)$ is connected, all cycles in each of the systems $\Xi'(R)$ and $\Xi''(R)$ must have the same degree, so that every cycle in $\bar{Z}(R)$ must split into two $\bar{R}$-disjoint parts. Of course, since in this case the hypothesis is preserved under any extension of the local domain R to another complete local domain dominating it, one sees readily that the point set $|\bar{Z}(R)|$ must in fact be absolutely connected.

We shall conclude this section with a remark on the uniqueness of the type of splittings which arise from the Hensel's Lemma. Let R be a valuation ring in a field K, with the residue field $\bar{R}$, and let $F(X)$ be a form in $R[X]$; if $F(X) = G(X)H(X)$ is a splitting of $F(X)$ in $R[X]$ (or in $K[X]$) such that $\bar{G}_R(X)$ and $\bar{H}_R(X)$ are relatively prime forms in $\bar{R}[X]$, then $G(X)$ and $H(X)$ are uniquely determined by the forms $\bar{G}_R(X)$ and $\bar{H}_R(X)$ up to proportionality factors in K. Therefore, if a, b, c are the *leading coefficients* (leading coefficient = coefficient of the term with the lowest "weight") in

$F(X)$, $G(X)$, $H(X)$ respectively, then the additional stipulation of the splitting $a = bc$ in R or K determines the forms $G(X)$ and $H(X)$ uniquely. For proof, one need only observe that since $\bar{G}_R(X)$ and $\bar{H}_R(X)$ are relatively prime, $G(X)$ and $H(X)$ must also be relatively prime, and the uniqueness of the splitting $F(X) = G(X)H(X)$ then follows from the unique factorization property of the polynomial ring $K[X]$. It is convenient to express this uniqueness by saying that the splitting $F(X) = G(X)H(X)$ is determined uniquely by the *initial condition* $(\bar{G}_R(X), \bar{H}_R(X); b, c)$.

5. Proof of Theorem 2. We shall first prove a lemma on a relation between a complete regular local ring and the completion of its quadratic valuation ring.

Let $(R, \mathfrak{p})$ be a complete regular local ring of dimension t, with the quotient field K and the residue field $\bar{R}$, and let $x_1, \cdots, x_t$ be the elements in a minimal basis for $\mathfrak{p}$. We denote by Q a system of representatives of $\bar{R}$ in R; apart from the condition that every element in $\bar{R}$ is the $\mathfrak{p}$-residue of exactly one element in Q, the elements in Q can be chosen arbitrarily, but for convenience we shall assume that Q contains the elements 0 and 1. We denote by $Q[x] = Q[x_1, \cdots, x_t]$ the set of all polynomials in $x_1, \cdots, x_t$ with coefficients in Q, and we shall say that an element in $Q[x]$ is *monic* if the coefficient of the leading term (i.e. the term with the lowest weight, each x_i being assigned the weight i) is 1. We denote by $Q(x)$ the set of all quotients $f(x)/g(x)$ of two elements in $Q[x]$ such that $g(x)$ is monic and that the polynomials $\bar{f}(T)$ and $\bar{g}(T)$ are relatively prime (where $T = (T_1, \cdots, T_t)$ is a system of indeterminates), and we shall say that an element $f(x)/g(x)$ in $Q(x)$ is homogeneous of degree i if both $f(x)$ and $g(x)$ are forms in x and $i =$ degre of $f(x)$ — degree of $g(x)$. We shall now associate with the system of elements $x = (x_1, \cdots, x_t)$ a system of independent variables $\boldsymbol{x} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_t)$ over $\bar{R}$ in Λ; then we can associate with each element $f(x)/g(x)$ in $Q(x)$ the element $\bar{f}(\boldsymbol{x})/\bar{g}(\boldsymbol{x})$ in the field $\bar{R}(\boldsymbol{x})$, which we shall call the *homogeneous residue* of $f(x)/g(x)$. It is easily seen that, by virtue of our choice of the elements in $Q(x)$ made above, this association of the elements in $Q(x)$ with their homogeneous residues in $\bar{R}(\boldsymbol{x})$ is a one-to-one mapping of $Q(x)$ onto $\bar{R}(\boldsymbol{x})$, which maps the subset $Q[x]$ onto the subring $\bar{R}[\boldsymbol{x}]$.

We denote by $Q[(x)]$ the set of all series of the type $\sum_{i=0}^{\infty} a^{(i)}$, where each term $a^{(i)}$ is a homogeneous element of degree i in $Q(x)$, and we denote by $Q[[x]]$ the subset in $Q[(x)]$ consisting of all such series in which each term

$a^{(i)}$ is a homogeneous degree i in $Q[x]$. Now, since R is a complete regular local ring, it can be easily shown that every element a in R can be represented *in a unique manner* by a series $a = \sum_{i=0}^{\infty} a^{(i)}$ in $Q[[x]]$, while conversely it is obvious that any series in $Q[[x]]$ is convergent in the topology of R and hence represent an element in R; this shows that we can set $R = Q[[x]]$, and we shall say that the series $\sum_{i=0}^{\infty} a^{(i)}$ is the *power series development* of the element a (with respect to the given choice of the basis elements $x_1, \cdots, x_t$ and the system Q), although it is in general not a power series in the usual sense. Similarly, if we denote by $(M, \mathfrak{q})$ the $\mathfrak{p}$-adic valuation ring of R, then every element in the completion M^* of M can be represented *in a unique manner* by a series in $Q[(x)]$, while conversely it is also obvious from the definition of the $\mathfrak{p}$-adic valuation that any series in $Q[(x)]$ is convergent in the topology of M and hence represent an element in M^*; therefore we can set $M^* = Q[(x)]$, and we shall say that the series in $Q[(x)]$ corresponding to an element in M^* is the *power series development* of this element.

If $a = \sum_{i=0}^{\infty} a^{(i)}$ is any element in M^*, then the first non-zero term $a^{(\mu)}$ in the series is called the *initial element* of a and the number μ is called the *order* of a; the homogeneous residue $\boldsymbol{a}^{(\mu)}$ of $a^{(\mu)}$ is also called the homogeneous residue of a and will be denoted also by $\boldsymbol{a}$. For elements of order zero in M^*, i.e. units in M^*, the mapping $a \to \boldsymbol{a}$ induces an isomorphism between the residue field $M^*/M^*\mathfrak{q}$ and the subfield in $\bar{R}(\boldsymbol{x})$ consisting of all homogeneous elements of order zero; we can therefore attach M^* to Λ by identifying $M^*/M^*\mathfrak{q}$ with this subfield, and we observe that with this attachment of M^* the homogeneous residue $\boldsymbol{a}$ of any element a of order zero coincides with its residue $\bar{a}$ modulo $M^*\mathfrak{q}$. Geometrically, we can consider the system $\boldsymbol{x} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_t)$ as the system of homogeneous coordinates of a generic point ζ over $\bar{R}$ of the center $\bar{x}(R)$ of the quadratic transform $x(R)$ of R, so that we can set $\bar{R}(\zeta) = M/\mathfrak{q} = M^*/M^*\mathfrak{q}$. However, we stress here that the elements $\boldsymbol{x}_i$ are not in $\bar{R}(\zeta)$, and that the symbol $\boldsymbol{x}$ denotes the system of t elements $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_t)$, not the point ζ, whose homogeneous coordinates are after all only determined up to a proportionality factor.

If $F(X)$ is a polynomial (or form) in $M^*[X] = M^*[X_0, X_1, \cdots, X_n]$, then we can consider the *power series development* $F(X) = \sum_{i=0}^{\infty} F^{(i)}(X)$, where $F^{(i)}(X)$ is a polynomial (or form) in X with coefficients which are homogeneous elements of degree i in $Q(x)$; the first non-vanishing term $F^{(\lambda)}(X)$

in this series is then called the *initial polynomial* (or *form*) of $F(X)$, and the polynomial (or form) $\boldsymbol{F}^{(\lambda)}(X)$ obtained from $F^{(\lambda)}(X)$ by replacing every coefficient in it by its homogeneous residue will be called the *homogeneous residue polynomial* (or *form*) of $F(X)$ and will be denoted also by $\boldsymbol{F}(X)$. it is clear that in case $F^{(0)}(X)$ does not vanish, we have the equation $\boldsymbol{F}(X) = \boldsymbol{F}^{(0)}(X) = \bar{F}(X)$.

Let η be a point of dimension $t-s$ over $\bar{R}$ in the center $\bar{x}(R)$ of the quadratic transform $x(R)$; without any loss of generality, we can assume that $\eta_1 = 1$ and that $\eta_{s+1}, \cdots, \eta_t$ are independent variables over $\bar{R}$. Since $\bar{x}(R)$ coincides with the projective space $\bar{S}^{t-1}$, there exist $s-1$ forms $\phi_2(X), \cdots, \phi_s(X)$ in $\bar{R}[X]$ of the same degree μ such that the elements $\phi_2(\zeta)/\zeta_1^{\mu}, \cdots, \phi_s(\zeta)/\zeta_1^{\mu}$ constitute a basis for the maximal prime ideal in the specialization ring $\mathfrak{Q}(\zeta/\eta, \bar{R})$ in $\bar{R}(\zeta)$ of $\bar{S}^{t-1}$ at η. If $f_2(X), \cdots, f_s(X)$ are forms in $R[X]$ such that $\bar{f}_i(X) = \phi_i(X)$, then the elements $x'_1 = x_1$, $x'_2 = f_2(x)/x_1^{\mu-1}, \cdots, x'_s = f_s(x)/x_1^{\mu-1}$ constitute a basis for the maximal prime ideal in $\mathfrak{Q}(x(R)/\eta)$. Since η is a specialization of ζ over $\bar{R}$, M is the quotient ring of $\mathfrak{Q}(x(R)/\eta)$ with respect to the prime ideal $\mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q}$, and since the ideal $\mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q}$ must then have the rank 1 and evidently contains the prime ideal generated by $x'_1 = x_1$ in $\mathfrak{Q}(x(R)/\eta)$, it must coincide with the latter. Furthermore, it is easily seen that $\mathfrak{Q}(x(R)/\eta)/\mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q}$ is isomorphic to $\mathfrak{Q}(\zeta/\eta, \bar{R})$ and hence can be identified with the latter, so that $M/\mathfrak{q} = \bar{R}(\zeta)$ is the quotient field of $\mathfrak{Q}(x(R)/\eta)/\mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q}$.

Consider now the completion $(R_1, \mathfrak{p}_1)$ of $\mathfrak{Q}(x(R)/\eta)$ and denote by $\mathfrak{q}_1$ the prime ideal generated by x'_1 in R_1 (note that x'_1 is also an element of a basis for $\mathfrak{p}_1$); since $\mathfrak{q}_1 = R_1(\mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q})$, it follows that $R_1/\mathfrak{q}_1$ is the completion of $\mathfrak{Q}(x(R)/\eta)/\mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q} = Q(\zeta/\eta, \bar{R})$. Let $(M', \mathfrak{q}')$ be the quotient ring of R_1 with respect to $\mathfrak{q}_1$, which is a real discrete valuation ring. Since M is the quotient ring of $\mathfrak{Q}(x(R)/\eta)$ with respect to $\mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q} = \mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q}'$, it can be embedded canonically as a subring in M', and if we attach M' to Λ compatibly with the attachement of M to Λ, then it is clear that M' dominates M. Consider now the completion M'^* of M'; it is important for our present purpose to observe that the completion M^* of M *can be embedded canonically in* M'^* *as a subspace.* In fact, according to [12], Lemma 7, we have the relation $R_1 \cap \mathfrak{q}'^i = R_1(\mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q}^i)$ for every positive integer i; it follows that

$$\mathfrak{Q}(x(R)/\eta) \cap (M \cap \mathfrak{q}'^i) = \mathfrak{Q}(x(R)/\eta) \cap (R_1 \cap \mathfrak{q}'^i)$$
$$= \mathfrak{Q}(x(R)/\eta) \cap R_1(\mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q}^i) = \mathfrak{Q}(x(R)/\eta) \cap \mathfrak{q}^i,$$

and hence $M \cap \mathfrak{q}'^i = \mathfrak{q}^i$. This shows that M' contains M as a subspace, and hence M'^* contains M^* as a subring.

Let $a = \sum_{i=\lambda}^{\infty} a^{(i)}$ be an element in M^*, expressed as a power series with $a^{(\lambda)} \neq 0$. We want to find the condition that this element a be contained in R_1; this question makes sense because both M^* and R_1 are subrings in M'^*. It is evidently sufficient to assume that every element $a^{(i)}$ is contained in $\mathfrak{Q}(x(R)/\eta)$; we shall show that this condition is also necessary. If a is contained in R_1, then a/x_1^λ is also contained in R_1; hence the $M'^*\mathfrak{q}'$-residue of a/x_1^λ, being also the $\mathfrak{q}_1$-residue of a/x_1^λ, is contained in the completion of $\mathfrak{Q}(\zeta/\eta, \bar{R})$. On the other hand, since a/x_1^λ and $a^{(\lambda)}/x_1^\lambda$ have the same $M'^*\mathfrak{q}'$-residue, and since the $M'^*\mathfrak{q}'$-residue of $a^{(\lambda)}/x_1^\lambda$, being also the $\mathfrak{q}$-residue of $a^{(\lambda)}/x_1^\lambda$, is contained in $\bar{R}(\zeta)$, it follows that the $M'^*\mathfrak{q}'$-residue of $a^{(\lambda)}/x_1^\lambda$ must be contained in both $\bar{R}(\zeta)$ and the completion of $\mathfrak{Q}(\zeta/\eta, \bar{R})$. Since $\bar{R}(\zeta)$ is the quotient field of $\mathfrak{Q}(\zeta/\eta, \bar{R})$, it follows from a well-known property of local rings that the $M'^*\mathfrak{q}'$-residue of $a^{(\lambda)}/x_1^\lambda$ must be contained in $\mathfrak{Q}(\zeta/\eta, \bar{R})$. Since this means that the $\mathfrak{q}$-residue of $a^{(\lambda)}/x_1^\lambda$ is contained in $\mathfrak{Q}(\zeta/\eta. \bar{R})$, it follows from the uniqueness of the power series development of a that $a^{(\lambda)}/x_1^\lambda$ must be contained in $\mathfrak{Q}(x(R)/\eta)$, and hence $a^{(\lambda)}$ itself must be contained in $\mathfrak{Q}(x(R)/\eta)$. Now since the element $a - a^{(\lambda)}$ is also an element in R_1, we can apply the same argument to it as we did to a and conclude that $a^{(\lambda+1)}$ must also be contained in $\mathfrak{Q}(x(R)/\eta)$. Thus, continuing in this way, we conclude that every element $a^{(i)}$ is contained in $\mathfrak{Q}(x(R)/\eta)$.

Since R is the intersection of all the rings $\mathfrak{Q}(x(R)/\eta)$ as η runs through all points in $\bar{x}(R)$ (in fact, it is sufficient to consider only point of dimension $t-2$ over $\bar{R}$), it follows that an element in M^* is in R if and only if it is contained in the completion of $\mathfrak{Q}(x(R)/\eta)$ for every point η (of dimension $t-2$ over $\bar{R}$) in $\bar{x}(R)$. We shall state this result as a lemma for later reference.

LEMMA 5.1. *An element in M^* is in R if and only if it is contained in the completion of $\mathfrak{Q}(x(R)/\eta)$ for every point η (of dimension $t-2$ over $\bar{R}$) in $\bar{x}(R)$.*

Consider now the canonical mapping of M'^* onto $M'^*/\mathfrak{q}'^* = M'/\mathfrak{q}'$, and let M_1 be the subring in M'^* consisting of all elements whose images under this mapping are in $R_1/\mathfrak{q}_1$; it is clear that M_1 contains $\mathfrak{q}'^*$ as a minimal prime ideal, and that $M_1/\mathfrak{q}'^* = R_1/\mathfrak{q}_1$ and $M_{1,\mathfrak{q}'^*} = M'^*$. In case $t = s = 2$, the local ring $R_1/\mathfrak{q}_1$ has the dimension 1 and hence is a complete real discrete valuation ring; it follows then that in this case M_1 is a discrete valuation ring of rank 2,

obtained from the composition of the valuations of M'^* and $R_1/\mathfrak{q}_1$, and that it is complete. This ring M_1 will be useful to us later; we observe here that M_1 dominates R_1.

We proceed now to prove Theorem 2.

Proof of Theorem 2. We observe first that, just as in the proof of the Principle of Degeneration in [4], we can without any loss of generality restrict ourselves to the case where Z is K-irreducible. For the condition for the supports of two positive cycles to have a common point can be expressed by a system of "universal" equations in the coefficients of their associated forms, and it is clear that such a condition is always preserved under a specialization. If Z is the sum of two positive cycles Z' and Z'', both rational over K, such that their supports meet, then the support of $\bar{Z}'(N)$ and $\bar{Z}''(N)$ must also meet for every N in $\mathfrak{V}(R)$, so that $\bar{Z}'(N)$ and $\bar{Z}''(N)$ cannot be $\bar{R}$-disjoint; it follows then that any $\bar{R}$-disjoint splitting of the system $\bar{Z}(R)$ will induce an $\bar{R}$-disjoint splitting of at least one of the two systems $\bar{Z}'(R)$ and $\bar{Z}''(R)$. Thus the validity of our theorem for every K-irreducible positive cycle will imply its validity in the general case.

Next, we show that we can reduce our theorem to the special case where Z is a positive 0-cycle. Let $u^{(i)}_j$, $i=1,\cdots,r$, $j=0,1,\cdots,n$, be $r(n+1)$ independent variables over K, and set $u^{(i)}=(u^{(i)}_0, u^{(i)}_1,\cdots,u^{(i)}_n)$, $i=1,\cdots,r$ (r being the dimension of the cycle Z). The ring $R[[u]]$ of all power series in $u^{(i)}_j$ with coefficients in R is easily seen to be a complete regular local ring (of dimension $t+r(n+1)$, t being the dimension of R), and we can attach it to Λ by identifying the residue field of $R[[u]]$ with $\bar{R}$, so that $R[[u]]$ dominates R. Let $F(U^{(0)},U^{(1)},\cdots,U^{(r)})$ be the associated form of Z, and consider the form $F(U^{(0)},u)=F(U^{(0)},u^{(1)},\cdots,u^{(r)})$ obtained from it by substituting the indeterminates $U^{(1)},\cdots,U^{(r)}$ by $u^{(1)},\cdots,u^{(r)}$; it is clear that since $F(U^{(0)},U^{(1)},\cdots,U^{(r)})$ is K-irreducible, $F(U^{(0)},u)$ must be $K(u)$-irreducible. Since the coefficients in $F(U^{(0)},u)$ are homogeneous of the same degree in the variables $u^{(i)}_j$, it is easily seen that $F(U^{(0)},u)$ must be irreducible also over the power series ring $K[[u]]$; and since $K[[u]]$ is a unique-factorization ring, this implies that $F(U^{(0)},u)$ is irreducible over the quotient field $K((u))$ of $K[[u]]$, and hence also irreducible over the quotient field $R((u))$ of $R[[u]]$. The form $F(U^{(0)},u)$ therefore determines a positive 0-cycle Z_u in S^n which is rational and irreducible over $R((u))$; and we observe that since Z_u is contained in the support of Z, the cycle $\bar{Z}_u(N_1)$ must be contained in the support of the cycle $\bar{Z}(N_1)$ for every valuation ring N_1 in $\mathfrak{V}(R[[u]])$. If $\bar{Z}(N)=\Omega'(N)+\Omega''(N)$ $(N\in\mathfrak{V}(R))$ is an $\bar{R}$-disjoint splitting of $\bar{Z}(R)$, then we denote by $\Omega'_u(N_1)$ and $\Omega''_u(N_1)$ the restrictions of

$\bar{Z}_u(N_1)$ in $|\Omega'(N_1)|$ and $|\Omega''(N_1)|$ respectively; the equation $Z_u(N_1) = \Omega'(N_1) + \Omega''_u(N_1)$ $(N_1 \in \mathfrak{V}(R[[u]]))$ then defines an $\bar{R}$-disjoint splitting of $\bar{Z}_u(N_1)$, provided it is not trivial for at least one N_1. But this is easily seen to be the case when we take for N_1 the quadratic valuation ring of $R[[u]]$; for in this case $\bar{u}^{(i)}(N_1)$, $i = 1, \cdots, r$, are independent generic points in S^n over a field of rationality for $\bar{Z}(N_1)$, and $\bar{Z}_u(N_1)$ is simply the intersection cycle of $\bar{Z}(N_1)$ with the linear subspace of dimension $n-r$ defined by the r dual hyperplanes of the points $\bar{u}^{(i)}(N_1)$. Thus the validity of our theorem for the 0-cycle Z_u implies its validity for the r-cycle Z.

Finally, we reduce our theorem to the case where the dimension t of R is equal to 2, whereby we note that the case $t = 1$ (Z being now a 0-cycle) reduces essentially to the Hensel's Lemma. We observe first that it is permissible to replace R by the complete local domain $R(v)^*$ obtained from R by the adjunction of $t-2$ independent variables $v = (v_1, \cdots, v_{t-2})$ over K; for the cycle Z will remain irreducible over $K(v)$, while any $\bar{R}$-disjoint splitting of $\bar{Z}(R)$ will induce an $\bar{R}(\bar{v})$-disjoint splittings of $\bar{Z}(R(v)^*)$ through the mapping $N \to N \cap K$ of $\mathfrak{V}(R(v)^*)$ into $\mathfrak{V}(R)$. We can therefore apply Lemma 2.9 (taking $d = t-2$) and consider the local domain R_v^* introduced there, which is a complete regular local ring of dimension 2; since K is algebraically closed in the quotient field of R_v^* and Z is irreducible over K, it follows that Z is also irreducible over the quotient field of R_v^*. On the other hand, since R_v^* dominates R, the system $\bar{Z}(R_v^*)$ is a subsystem of $\bar{Z}(R)$; it follows that an $\bar{R}$-disjoint splitting of $\bar{Z}(R)$ will induce also such a splitting of $\bar{Z}(R_v^*)$, which must be of course also $\bar{R}(\bar{v}, \bar{u})$-disjoint, using the notation in Lemma 2.9. Thus the validity of our theorem for the case $r = 0$, $t = 2$, will imply its validity in the general case.

We therefore have to prove our theorem only for the case where Z is a K-irreducible positive 0-cycle and R is a complete regular local ring of dimension 2; we assume that there exists an $\bar{R}$-disjoint splitting of the system $\bar{Z}(R)$ and we shall show that the cycle Z must then be reducible over K, in contradiction to our hypothesis. In terms of the associated form $F(U)$, the given $\bar{R}$-disjoint splitting of $\bar{Z}(R)$ can be expressed by a splitting $\bar{F}_N(U) \doteq \Phi_N(U)\Theta_N(U)$ of the system of forms $\bar{F}_R(U)$ into two systems $\Phi_R(U)$ and $\Theta_R(U)$ such that $\Phi_N(U)$ and $\Theta_N(U)$ are relatively prime over $\bar{R}$ for every valuation ring N in $\mathfrak{V}(R)$, and our theorem will clearly follow if we show that there is then a splitting $F(U) = G(U)H(U)$ in $R[U]$ such that $\bar{G}_N(U) \doteq \Phi_N(U)$ and $\bar{H}_N(U) \doteq \Theta_N(U)$ for every N in $\mathfrak{V}(R)$. Let y be the associated point of Z, and consider the rational transform $y(R)$; since R is a regular local ring of dimension 2, there exists according to Theorem 1, Corollary 1, a finite quadratic transform $z(R)$ of a certain order π

(along a suitable sequence of valuations) such that the correspondence from $z(R)$ to $y(R)$ is everywhere regular. In case $\pi = 0$, i.e., $z(R) = R$, the center $\bar{y}(R)$ is a uniquely determined point and the corresponding form is just the residue form $\bar{F}(U)$; it follows that $\bar{F}_N(U) = \bar{F}(U)$ for every N in $\mathfrak{V}(R)$, so that all forms in the system $\bar{F}_R(U)$ are essentially the same. Now, there are only a finite number of ways in which the form $\bar{F}(U)$ can split into two factors and the set of all N such that the splitting $\bar{F}_N(U) = \Phi_N(U)\Theta_N(U)$ coincides with a given splitting of $\bar{F}(U)$ is easily seen to be a closed subset in $\mathfrak{V}(R)$. Since according to Theorem 1, Corollary 2, the space $\mathfrak{V}(R)$ is connected, it follows that all splittings in the given $\bar{R}$-disjoint splitting of the system $\bar{F}_R(U)$ must coincide with one single splitting $\bar{F}(U) \doteq \Phi(U)\Theta(U)$; furthermore, since $\bar{F}(U)$ is rational over $\bar{R}$ and the splitting is $\bar{R}$-disjoint, it follows that both $\Phi(U)$ and $\Theta(U)$ must be rational over $\bar{R}$. By Hensel's Lemma for the local ring R, there exist forms $G(U)$ and $H(U)$ in $R[U]$ such that $F(U) = G(U)H(U)$, $\bar{G}(U) \doteq \Phi(U)$, and $\bar{H}(U) \doteq \Theta(U)$. This proves our assertion for $\pi = 0$. We shall prove our assertion in the general case by induction on the order π of $z(R)$; we therefore assume that our assertion has already been proved for the case where $y(R)$ is an everywhere regular transform of a finite quadratic transform of R of order less than π.

It is well-known that R is a unique -factorization ring; for convenience, we shall assume that the coefficients in $F(U)$ have been so chosen that they are all in R, but without any common factor. Consider the homogeneous residue form $\boldsymbol{F}(U)$ of $F(U)$; since $\boldsymbol{F}(U)$ differs from $\bar{F}_M(U)$ only by a proportionality factor in $\bar{R}(\boldsymbol{x})$ (we recall that $M = Q(x(\mathrm{R})/\zeta)$ and ζ is the point in $\bar{x}(R)$ with $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_t)$ as homogeneous coordinates), the given splitting $\bar{F}_M(U) \doteq \Phi_M(U)\Theta_M(U)$ induces a splitting $\boldsymbol{F}(U) \doteq \Phi(U)\Theta(U)$ in $\bar{R}[\boldsymbol{x}]$, with $\Phi(U) \doteq \Phi_M(U)$ and $\Theta(U) \doteq \Theta_M(U)$. We assume now that (A) *the leading coefficient α in $\boldsymbol{F}(U)$ does not vanish*; if we denote by δ the highest common factor (in $\bar{R}[\boldsymbol{x}]$) of all coefficients in $\boldsymbol{F}(U)$, and choose the forms $\Phi(U)$ and $\Theta(U)$ (which are each determined up to a proportionality factor in $\bar{R}(\boldsymbol{x})$) so that the coefficients of each form are all in $\bar{R}[\boldsymbol{x}]$ but without any common factor, then we have $\boldsymbol{F}(U) = \delta\Phi(U)\Theta(U)$ up to a proportionality factor in $\bar{R}$, which can be absorbed in δ. If β and γ are the leading coefficients in $\Phi(U)$ and $\Theta(U)$ respectively, we have then $\alpha = \delta\beta\gamma$; we assume that (B) *the elements δ, β, γ are relatively prime.* The properties (A) and (B) assumed here, if not already present, can be easily obtained as follows. Let $w = (w_0, w_1, \cdots, w_n)$ be a system of independent variables over K, and consider the complete local ring $R(w)^*$, which is also regular and has the dimension 2, attached to Λ by identifying the elements $\bar{w} = (\bar{w}_0, \bar{w}_1, \cdots, w_n)$ with a system of independent variables over $\bar{R}$. We apply to $F(U)$ the linear

transformation which leaves $U_1, \cdots, U_n$ invariant but carries U_0 into $\sum_{i=0}^{n} w_i U_i$, and apply to $\boldsymbol{F}(U)$, $\Phi(U)$, $\Theta(U)$ the corresponding transformation whereby $U_1, \cdots, U_n$ remain invariant but U_0 is carried into $\sum_{i=0}^{n} \bar{w}_i U_i$; then the elements $\boldsymbol{F}(\bar{w})$, $\Phi(\bar{w})$, and $\Theta(\bar{w})$ becomes the leading coefficients of the transformed forms of $\boldsymbol{F}(U)$, $\Phi(U)$, and $\Theta(U)$ respectively. It is clear that our problem is invariant under such a linear transformation. Now, it is obvious that (A) holds for the transformed forms; while as to (B), we need only to observe that any common factor (in $\bar{R}[\boldsymbol{x}]$) of all coefficients in $\boldsymbol{F}(U)$ appears with the minimum power in $\boldsymbol{F}(\bar{w})$ (so that we can assume δ to be relatively prime to β and γ), and that since $\Phi(U)$ and $\Theta(U)$ are relatively prime with their coefficients determined as indicated above, the elements $\Phi(\bar{w})$ and $\Theta(\bar{w})$ must be relatively prime (so that we can assume β and γ to be relatively prime). On the other hand, since the field K is algebraically closed in the quotient field of $R(w)^*$, any splitting $F(U) = G(U)H(U)$ of $F(U)$ in $R(w)^*[U]$ must be already a splitting in $R[U]$, so that the validity of our assertion for $R(w)^*$ will imply its validity for R. Thus we have shown that we can assume the properties (A) and (B) without any loss of generality.

Let λ be the order of $F(U)$ in its power series development as a form in $M^*[U]$, and let a be the leading coefficient $F(U)$; by (A), the order of a must also be λ and we have $\boldsymbol{a} = \alpha = \beta\gamma\delta$, so that λ is also the degree of α (as a form in $\bar{R}[\boldsymbol{x}]$). Let μ, ν, ϵ be the degrees of β, γ, δ respectively, so that $\lambda = \mu + \nu + \epsilon$.

We now invoke a property of the local ring R which is dependent on its dimension being 2. According to [8], Satz 9 (see item (b) in the proof), any splitting of $\boldsymbol{a}$ into relatively prime elements is induced by a corresponding splitting of a in R. Since β, γ, δ are relatively prime, there exist therefore elements b, c, d in R such that $a = bcd$, $\boldsymbol{b} = \beta$, $\boldsymbol{c} = \gamma$, $\boldsymbol{d} = \delta$. Let $\delta = \delta_1 \cdots \delta_s$ be the factorization of δ into powers of distinct prime elements in $\bar{R}[\boldsymbol{x}]$; there exist then elements $d_1, \cdots, d_s$ in R such that $d = d_1 \cdots d_s$, $\boldsymbol{d}_1 = \delta_1, \cdots, \boldsymbol{d}_s = \delta_s$. We denote by ϵ_i the order of d_i, which is the degree of δ_i, so that $\epsilon = \epsilon_1 + \cdots + \epsilon_s$. We observe that since each δ_i is a power of an irreducible form in $\bar{R}[\boldsymbol{x}]$, it determines a point $\xi^{(i)}$ in $\bar{x}(R)$, unique up to conjugates over $\bar{R}$, which is algebraic over $\bar{R}$, so that $Q(x(R)/\xi^{(i)})$ has the dimension 2.

Let η be a point in $\bar{x}(R)$; following the notations introduced in the proof of Lemma 5.1, we shall denote the completion of $Q(x(R)/\eta)$ by R_1, and we observe that R_1 is now a complete regular local ring of dimension $\leqq 2$ (the dimension is equal to 1 if and only if η is a generic point of $\bar{x}(R)$ over $\bar{R}$). Let $z'(R_1)$ be the finite quadratic transform of R_1 along the subsequence of

those valuations in the definition of $z(R)$ which dominate $Q(x(R)/\eta)$, or rather the sequence of the extensions of these (real discrete) valuations to R_1; it is easily seen that the correspondence from $z'(R_1)$ to $z(R_1)$ is regular everywhere, from which it follows that the correspondence from $z'(R_1)$ to $y(R_1)$ is also regular everywhere. Since the order of the quadratic transform $z'(R_1)$ is evidently less than π, we can apply our induction hypothesis. We recall that if N is any valuation ring dominating R, then by definition $\Phi_N(U) = \Phi_{K \cap N}(U)$ and $\Theta_N(U) = \Theta_{K \cap N}(U)$, and that $\Phi_{R_1}(U)$ and $\Theta_{R_1}(U)$ are subsystems of $\Phi_R(U)$ and $\Theta_R(U)$ respectively obtained by restricting N to valution rings dominating R_1. It is clear that the equation $\bar{F}_N(U) = \Phi_N(U)\Theta_N(U)$, for every N dominating R_1, defines a splitting of $\bar{F}_{R_1}(U)$ into the two systems $\Phi_{R_1}(U)$ and $\Theta_{R_1}(U)$, which is clearly $\bar{R}$-disjoint and hence also $\bar{R}(\eta)$-disjoint. By induction hypothesis, there exists therefore a splitting $F(U) = G_\eta(U)H_\eta(U)$ in $R_1[U]$ such that $G_\eta(U)$ generates the system $\Phi_{R_1}(U)$ and $H_\eta(U)$ generates the system $\Theta_{R_1}(U)$. Now, this splitting $F(U) = G_\eta(U)H_\eta(U)$ in $R_1[U]$ implies a splitting of the leading coefficient a in $F(U)$ into two factors in R_1, which we shall now proceed to investigate. Without any loss of generality, we can assume for convenience that $\eta_1 \neq 0$, as we did in the proof of Lemma 5.1; then all coefficients in $F(U)$ are divisible by x_1^λ in R_1, and since R_1 is a unique factorization ring, there exist positive integers λ' and λ'' ($\lambda' + \lambda'' = \lambda$) such that all coefficients in $G_\eta(U)$ are divisible by $x_1^{\lambda'}$ and all coefficients in $H_\eta(U)$ are divisible by $x_1^{\lambda''}$. If we set $F^0(U) = (1/x_1)^\lambda F(U)$, $G^0(U) = (1/x_1)^{\lambda'} G_\eta(U)$, and $H^0(U) = (1/x_1)^{\lambda''} H_\eta(U)$, then we have evidently the splitting $F^0(U) = G^0(U)H^0(U)$ in $R_1[U]$. We set $a^0 = (1/x_1)^\lambda a$, $b^0 = (1/x_1)^\mu b$, $c^0 = (1/x_1)^\nu c$, $d^0 = (1/x_1)^\epsilon d$, and $d_i^0 = (1/x_1)^{\epsilon_i} d_i$; then the elements a^0, b^0, c^0, d^0, d_i^0 are all in R_1, and a^0 is the leading coefficient in $F^0(U)$. The situation divides into two cases, which we shall treat separately.

(1) The point η coincides with one of the points $\xi^{(1)}, \cdots, \xi^{(s)}$, say $\xi^{(j)}$. In this case, it is easily seen that the elements b^0, c^0, d_i^0 $(i \neq j)$ are all units in R_1; for example, the p_1-residue of b^0 is the value of β for $\boldsymbol{x}_1 = 1$, $\boldsymbol{x}_2 = \eta_2/\eta_1$, which is not zero since β and δ_j are relatively prime in $\bar{R}[\boldsymbol{x}]$. The splitting $F^0(U) = G^0(U)H^0(U)$ induces therefore a splitting of d_j^0 into two factors in R_1. We maintain that this splitting of d_j^0 is induced by a splitting $d_j = d'_j d''_j$ in R; in other words, if ϵ' and ϵ'' are the orders of d'_j and d''_j respectively, then the given splitting of d_j^0 coincides with the splitting $d_j^0 = (d'_j/x_1^{\epsilon'})(d''_j/x_1^{\epsilon''})$ up to units in R_1. To prove this, it is sufficient to show that if e is a prime factor of d_j, of order ρ in M, then $e^0 = e/x_1^\rho$ must be a prime element in R_1; for we can then apply this argument to every

prime factor of d_j and obtain a complete factorization of d_j^0 in R_1, and our assertion then follows from the unique factorization property of R_1. First, we note that e^0 is a prime element in $R[x_2/x_1]$; for, if e^0 splits into two factors in $R[x_2/x_1]$, they must be of the form $e'/x_1^{\rho'}$ and $e''/x_1^{\rho''}$ with e' and e'' in R, and again the unique factorization property in R shows that we must have $\rho = \rho' + \rho''$ and hence $e = e'e''$. Next, since $\mathfrak{Q}(x(R)/\eta)$ is the quotient ring of $R[x_2/x_1]$ with respect to a prime ideal which contains the element e^0, it follows from a well-known property of quotient ring that e^0 must remain a prime element in $Q(x(R)/\eta)$. Now, consider the residue rings R/Re and $Q(x(R)/\eta)/Q(x(R)/\eta)e^0$; since $Q(x(R)/\eta)e^0 \cap R = R[x_2/x_1]e^0 \cap R = Re$, we can embed the former canonically in the latter. It is clear that both rings are local domains of dimension 1, and that their residue fields can be identified with $\bar{R}$ and $\bar{R}(\eta)$ respectively. Since R is complete, R/Re is also complete, and since $\bar{R}(\eta)$ is algebraic over $\bar{R}$, it follows ([6], Theorem 8) that $Q(x(R)/\eta)/Q(x(R)/\eta)e^0$ is a finite module over R/Re and hence ([3], § III, Proposition 8) must also be complete. This implies that

$$Q(x(R)/\eta)/Q(x(R)/\eta)e^0 = R_1/R_1e^0,$$

so that R_1/R_1e^0 is a local domain; this shows that e^0 is a prime element in R_1. It is clear that since $d'_j/x_1^{\epsilon'}$ and $d''_j/x_1^{\epsilon''}$ coincide up to units in R_1 with the leading coefficients in $G^0(U)$ and $H^0(U)$ respectively, we can take as the respective leading coefficients in the forms $G_\eta(U)$ and $H_\eta(U)$ any multiples of d'_j and d''_j in R which are factors in a splitting of a in R.

(2) The point η is distinct from the points $\xi^{(1)}, \cdots, \xi^{(s)}$. In this case, the form $\boldsymbol{F}(U)$ does not vanish identically at the point η, and hence at least one coefficient in $F^0(U)$ must be a unit in R_1; and this implies that also both $G^0(U)$ and $H^0(U)$ must have at least one coefficient which is a unit in R_1. If we denote by $\bar{F}^0(U)$, $\bar{G}^0(U)$, and $\bar{H}^0(U)$ the respective residue forms in $\bar{R}_1[U]$, then we must have $F^0{}_N(U) \doteq \bar{F}^0(U)$, $G^0{}_N(U) \doteq \bar{G}^0(U)$, and $H^0{}_N(U) \doteq \bar{H}^0(U)$ for every valuation ring N dominating R_1; it follows that $\Phi_N(U) \doteq \bar{G}^0(U)$ and $\Theta_N(U) \doteq \bar{H}^0(U)$ for every N dominating R_1, and hence in particular $\Phi_{M_1}(U) \doteq \bar{G}^0(U)$ and $\Theta_{M_1}(U) \doteq \bar{H}^0(U)$, where M_1 is the valuation ring of rank 2 introduced after Lemma 5.1. On the other hand, since M'^* is a quotient ring of M_1, $\Phi_{M_1}(U)$ and $\Theta_{M_1}(U)$ are specializations of $\Theta_{M'^*}(U) = \Phi_M(U)$ and $\Phi_{M'^*}(U) = \Theta_M(U)$ respectively over the specialization $\zeta \to \eta$ over $\bar{R}$; it follows that $\bar{G}^0(U)$ and $\bar{H}^0(U)$ are up to proportionality factors equal to the forms obtained from $\Phi(U)$ and $\Theta(U)$ respectively by setting $(x_1, x_2) = (\eta_1, \eta_2)$. Now, since β and γ have no common factor in $\bar{R}[\boldsymbol{x}]$, at least one of them does not vanish at the point η. If β does not vanish at η, then the leading coefficient in $\bar{G}^0(U)$ does not vanish and hence

the leading coefficient in $G^0(U)$ must be a unit in R_1; since in this case the elements b and d^0 are units in R_1, the leading coefficient in $H^0(U)$ must be equal to c up to a unit in R_1. On the other hand, if γ does not vanish at η, then the same argument with $G^0(U)$ and $H^0(U)$ interchanged shows that c^0 and d^0 are units in R_1, and that the leading coefficient in $G^0(U)$ must be equal to b up to a unit in R_1. Going over to the forms $G_\eta(U)$ and $H_\eta(U)$, we conclude that wec an take as their respective leading coefficients any multiples of b and c in R which are factors in a splitting of a in R.

We set $d' = d'_1 \cdot \cdots \cdot d'_s$ and $d'' = d''_1 \cdot \cdots \cdot d''_s$, so that we have $d = d'd''$, and consider the splitting $a = bd' \cdot cd''$ in R. We shall now normalize the forms $G_\eta(U)$ and $H_\eta(U)$, for every point η in $\bar{x}(R)$, by taking bd' as the leading coefficient in $G_\eta(U)$ and cd'' as the leading coefficient in $H_\eta(U)$. This choice of the leading coefficients is permissible in view of what we have just shown above. For $\eta = \zeta$, we obtain a splitting $F(U) = G_\zeta(U)H_\zeta(U)$ in $M^*[U]$; incidentally, we observe that this is the splitting according to Hensel's Lemma with the intial conditions $(\Phi_M(U), \Theta_M(U); bd', cd'')$, in the terminology introduced at the end of the section 4. We shall now show that $G_\zeta(U) = G_\eta(U)$ and $H_\zeta(U) = H_\eta(U)$ for every point η in $\bar{x}(R)$, in the sense that they are the same forms in $M'^*[U]$, so that all the splittings $F(U) = G_\eta(U)H_\eta(U)$ are in reality one and the same. This will prove our theorem. For, then the forms $G_\zeta(U)$ and $H_\zeta(U)$ must be both in $R[U]$ according to Lemma 5.1 (applied to every coefficient in them); if we set $G(U) = G_\zeta(U)$ and $H(U) = H_\zeta(U)$, then $F(U) = G(U)H(U)$ is a splitting in $R[U]$ and it is clear that $\bar{G}_N(U) \doteq \Phi_N(U)$ and $\bar{H}_N(U) \doteq \Phi_N(U)$ for every valuation ring N in K dominating R.

It remains therefore to show that $G_\zeta(U) = G_\eta(U)$ and $H_\zeta(U) = H_\eta(U)$, for every point η in $\bar{x}(R)$, whereby we shall use again the notations introduced in the proof of Lemma 5.1. Consider the valuation ring M_1 of rank 2 introduced after Lemma 5.1. It is clear that

$$\bar{G}_{\zeta,M'^*}(U) \doteq \bar{G}_{\zeta,M}(U) \doteq \Phi_M(U) \doteq \Phi_{M'^*}(U)$$

and

$$\bar{H}_{\zeta,M'^*}(U) \doteq \bar{H}_{\zeta,M}(U) \doteq \Theta_M(U) \doteq \Theta_{M'^*}(U);$$

since M'^* is a quotient ring of M_1, it follows that both $\bar{G}_{\zeta,M_1}(U)$ and $\Phi_{M_1}(U)$ are specializations of $\Phi_M(U)$ over the specialization $\zeta \to \eta$ over $\bar{R}$, and that both $\bar{H}_{\zeta,M_1}(U)$ and $\Theta_{M_1}(U)$ are specializations of $\Theta_M(U)$ over the specialization $\zeta \to \eta$ over $\bar{R}$. Since $\bar{x}(R)$ is a non-singular curve, the specializations of both $\Phi_M(U)$ and $\Theta_M(U)$ over the specialization $\zeta \to \eta$ over $\bar{R}$ must be unique; it follows that $\bar{G}_{\zeta,M_1}(U) \doteq \Phi_{M_1}(U)$ and $\bar{H}_{\zeta,M_1}(U) \doteq \Theta_{M_1}(U)$. On the other hand, by the inductive hypothesis, we have $\bar{G}_{\eta,M_1}(U) \doteq \Phi_{M_1}(U)$ and $\bar{H}_{\eta,M_1}(U)$

$\doteq \Theta_{M_1}(U)$. This shows that $\bar{G}_{\zeta,M_1}(U) \doteq \bar{G}_{\eta,M_1}(U)$ and $\bar{H}_{\zeta,M_1}(U) \doteq \bar{H}_{\eta,M_1}(U)$. Recalling that the forms $G_\zeta(U)$ and $G_\eta(U)$ have the same leading coefficient bd', and that the forms $H_\zeta(U)$ and $H_\eta(U)$ have the same leading coefficient cd'', we see that the two splittings $F(U) = G_\zeta(U)H_\zeta(U)$ and $F(U) = G_\eta(U)H_\eta(U)$ in $M_1[U]$ have the same initial condition

$$(\bar{G}_{\zeta,M_1}(U), \bar{H}_{\zeta,M_1}(U)\,; bd', cd'') = (\bar{G}_{\eta,M_1}(U), \bar{H}_{\eta,M_1}(U)\,; bd', cd'')\,;$$

it follows from the remark at the end of section 4 that $G_\zeta(U) = G_\eta(U)$ and $H_\zeta(U) = H_\eta(U)$. This concludes the proof of Theorem 2.

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